Partnership with Persistence

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Abstract

We study a continuous-time model of partnership with persistence and imperfect state monitoring. Partners exert private efforts to shape the stock of fundamentals, which drives the profits of the partnership, and the profits are the only public signal. The near-optimal strongly symmetric equilibria are characterized by a novel differential equation that describes the supremum of equilibrium incentives for any level of relational capital. Under (almost) perfect monitoring of the fundamentals, the only equilibria are (approximately) stationary Markov. Imperfect monitoring helps sustain relational incentives and increases the partnership's value by extending the relevant time horizon for incentive provision. The results are consistent with the predominance of partnerships and relational incentives in environments where effort has long-term and qualitative impact and in which progress is hard to measure.

Keywords: partnership, dynamic games, continuous time, relational incentives

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1 Introduction

Partnerships are among the main forms of organizing joint economic activity. Characterized by common ownership, which ties the partners in an ongoing, long-term relationship, partnerships are common among individuals, businesses, and constitute one of the dominant forms of structuring a firm—along with corporations and limited liability companies. As any organization, partnerships face an incentive problem of motivating members to exert effort and, hence, to contribute to its success.

The ongoing, dynamic nature of partnerships complicates the incentive problem. To fix ideas, consider a start-up. On a daily basis, each partner devotes her effort to improving the venture's fundamentals: upgrading the quality of the product; broadening the customer base; facilitating access to external capital; improving the internal organization; and more. Each of these fundamentals evolves over time, affected by the partners' efforts and by the circumstances. Moreover, none of the fundamentals needs to be directly observed by the partners, who see only how they are gradually reflected in the shared profits, customer reviews, or internal audits. However, a long-term partnership also offers a unique advantage: it fosters relational incentives. A partner has incentives to work hard not only to boost profits, but also to boost observable outcomes, morale, and, ultimately, to increase the future effort choices of her partners.

In this paper, we analyze the effect of imperfect monitoring of the fundamentals on the provision of relational incentives in a partnership. In the continuous-time model we consider, stochastic fundamentals are a persistent state, shaped by partners' efforts. Our main result is that worse monitoring of the venture delays information about partners' efforts, which may improve the provision of relational incentives and, hence, increase the partnership value. The results are consistent with the predominance of partnerships and relational incentives in environments where effort has long-term and qualitative impact and in which progress is hard to measure. On a technical side, we develop a method to solve for near-optimal strongly symmetric equilibria of the partnership. It extends the stochastic control techniques to a wide class of games with persistence and imperfect state monitoring.

In our continuous-time model, at any point in time, two partners privately choose costly effort and evenly split the profits of their venture. Fundamentals change stochastically, driven by the sum of efforts and, in turn, equal the expected profit flow. The private marginal benefit of effort due to the direct effect on profits (Markov incentives) is constant and equals half of the social marginal benefit of effort. This results in the unique, stationary Markov equilibrium (Proposition 1). In the model, neither efforts nor fundamentals are observable, and profits, which follow a Brownian diffusion, are the partners' only publicly available information.

Our minimal monitoring structure does not allow the signals to separately identify each partner's effort (Fudenberg et al. [1994]). Consequently, we focus on the strongly symmetric equilibria (SSE) and relational incentives, with partners coordinating on relatively efficient (inefficient) effort after outcomes indicative of high (low) effort.

The results in this paper rely on the fact that the persistent effect of effort combined with the imperfect state monitoring lengthen the time horizon for incentive provision. If the current profits depend only on current efforts—in the i.i.d., repeated game setting—or if fundamentals are perfectly monitored—in a stochastic game setting—the rewards must be provided instantaneously. Increased effort brings about unexpectedly high profits (the signals that partners rely on) only in the same period, with fundamentals updated instantly in the next. Outside of those limiting environments, signals indicating increased effort today are spread over time. Poor monitoring results in slow learning about the fundamentals, and relatively more information coming in late.

The longer time horizon for incentive provision may, in turn, facilitate the provision of relational incentives. The reason is that relational rewards must take a form of a promise of improved future relationship. If the relationship is already at the bliss point, then immediate relational rewards are unavailable. Indeed, in a Brownian diffusion model like ours, no immediate incentives can be provided at the bliss point.[1] We show that when monitoring is (near-)perfect and horizon for incentives short, relational incentives (nearly) unravel (Proposition 4). However, with poor monitoring partners work at a bliss

[1]Sannikov and Skrzypacz [2007] show that the impossibility persists in discrete-time models with short period lengths.

point not for immediate rewards, but to invest in good outcomes and in an improved relationship in the future.

The benefit of poor monitoring on relational incentives and partnerships has implications for the structure of a firm in different informational environments. Poor monitoring always hurts the provision of incentives, if employees are rewarded either by better reputation and competitive wage (career concerns, Holmström [1999]), or by contracts that provide performance-based payments. In the first case, poor monitoring delays the arrival of reputational rewards, while with contracts it increases information asymmetry and the cost of providing incentives. A cross-sectional implication is that partnerships are favored and, hence, more prevalent, in environments in which the effects of effort are hard to measure or quantify. Moreover, performance data becoming better and cheaper is more advantageous to corporations and other organizations that rely on incentives that are based on pecuniary rewards.

Finally, relational incentives are hurt when fundamentals are more stochastic, or when there is more uncertainty about the stochastic quality of the venture, as in the case of young enterprises (Proposition 5). While young ventures may provide incentives based on the manipulation of public beliefs, either about oneself (career concerns) or the productivity of the venture (e.g., encouragement effect, Bolton and Harris [1999]), more established ones must rely on relational incentives and, hence, on the mechanism described in this paper. We show that an established partnership may unravel as a consequence of a short spat of bad outcomes, with hardly any effect on its expected productivity or profitability, or quality of the partners (Corollary 2, "Beatles' Break-up".)

To analyze relational incentives in a setting with persistence and imperfect state monitoring, we develop a novel method. It is based on characterizing the upper boundary of relational incentives achievable in a local SSE, under only local incentive constraints, as a function of expected value of future efforts (relational capital, an equivalent of continuation value in an i.i.d. setting). Theorem 1 shows that the boundary of incentives satisfies an appropriate ordinary differential HJB equation and provides boundary conditions. The right-most argument characterizes the optimal relational capital and partnership's value.

Theorem 2 shows, roughly, that a modified boundary is self-generating (as in [Abreu et al., 1990]) and defines a near-optimal local SSE. It is a convenient tool for analyzing the dynamics of effort (Section 3.2 discusses "rallying" and "coasting" in a partnership).

The novel approach of maximizing incentives as a function of value, rather than vice-versa, requires extending the stochastic control methods. It results in the law of motion of the relational capital (state variable) depending on the level of the objective function (incentives), via the effort chosen. This dependence is not allowed in stochastic control, yet we verify that the HJB characterization of the boundary is valid.[2] Another difficulty is familiar: In Theorem 3, we provide conditions on the primitives so that the constructed strategies are not only locally, but fully incentive-compatible. Finally, to highlight the portability of our method, we provide a general model and the HJB characterization (Appendix C) and discuss application to models of capital accumulation, oligopoly, and team production with asymmetric players (Appendix C.1).

1.1 Related Literature

This paper belongs to the literature on free-riding in groups, in dynamic environments.[3] The repeated partnership game was first studied by Radner [1985] and Radner et al. [1986], who demonstrate inefficiency of equilibria, and by Fudenberg et al. [1994], who pin down the identifiability conditions violated in the model. While symmetric equilibria feature a "bang-bang" property (Abreu et al. [1986]), in our case signals about effort accrue slowly over time and result in gradual equilibrium dynamics.

Abreu et al. [1991], Sannikov and Skrzypacz [2007, 2010] show how frequent interactions may have a detrimental effect on incentives. In particular, the discrete-time approximation of a Brownian model of partnership or collusion in Sannikov and Skrzypacz [2007], which has either no persistence or a perfectly monitored state, has no relational

[2] Relatedly, the results in Sannikov [2007] and Faingold and Sannikov [2020] extend stochastic control results to settings in which the law of motion of state variables depends not on the level, but on the derivative of the value function.

[3] See Olson [1971], Alchian and Demsetz [1972], Holmstrom [1982], as well as Legros and Matthews [1993] and Winter [2004] for seminal contributions in static settings.

incentives.[4] Faingold and Sannikov [2011] and Bohren [2018] establish related results with one long-lived player in a competitive market setting. We show that the impossibility is not inherent to continuous-time modeling, but is a consequence of the monitoring structure instead.[5] Rahman [2014] shows how relational incentives may be restored in the presence of a mediator, using secret monitoring and infrequent coordination.

Our paper ties into the literatures on career concerns (see Holmström [1999] and Cisternas [2017]) and on experimentation in teams (see Bolton and Harris [1999], Georgiadis [2014], and Cetemen et al. [2017] for Brownian models like ours).[6] In career concerns models, equilibrium play depends only on beliefs about an exogenous state;[7] the literature on experimentation in teams studies effects of payoff or information externalities on incentives and focuses on Markov equilibria as well, with no relational component. Our paper is complementary: It has production technology independent of history (as in Holmström [1999]), with constant Markov incentives, but we focus on optimal equilibria, which rely on relational incentives. Our equilibrium characterization is equally tractable, with incentives driven by the endogenous relational capital of the partnership.[8]

Persistence plays an important role in dynamic contracting models with learning (see Jarque [2010], Williams [2011], Prat and Jovanovic [2014], Sannikov [2014], Prat [2015], DeMarzo and Sannikov [2016], and He et al. [2017] for Brownian models like ours). Although the questions and the incentive mechanisms are different from ours, the literature has long recognized the difficulty of accounting for the marginal benefits of deviations, or incentives, and of verifying global incentive compatibility. Our solution method for a game setting, in which every player requires incentives, is new and is based on maximizing

[4] See, also, Sadzik and Stacchetti [2015] for the discrete-time approximation of the Brownian Principal-Agent, rather than partnership model.

[5] Quick learning and ratchet effect also prevents nontrivial effort in Bhaskar [2014], but for entirely different reasons. There, impossibility relies on a setting that combines continuous and discrete choices.

[6] See, among others, Keller et al. [2005], Keller and Rady [2010], Klein and Rady [2011], and Bonatti and Hörner [2011] for experimentation in teams with exponential bandit models. See, also, Décamps and Mariotti [2004], Rosenberg et al. [2007], Murto and Välimäki [2011], and Hopenhayn and Squintani [2011] for related stopping games with incomplete information.

[7] Specifically, Cisternas [2017] provides a differential equation also for the stock of incentives, just as in this paper, but in a differentiable Markov equilibrium, as a function of public beliefs about the state.

[8] In our near-optimal equilibria, working to rally the partnership is related to the encouragement effect identified by Bolton and Harris [1999], and coasting is reminiscent of the work-shirk-work dynamics in the reputation model of Board and Meyer-ter Vehn [2013].

incentives, rather than on including them as an additional state variable. Moreover, we provide conditions on the primitives of the model (in our case, the convexity of costs), so that the solution of the relaxed problem is fully incentive-compatible (see Edmans et al. [2012] and Cisternas [2017] for related results).[9]

Finally, we interpret our results as providing a rationale for the prevalence of partnerships in industries with poor monitoring of the venture's progress. In particular, as documented by Von Nordenflycht [2010], "opaque" quality is a key characteristic of the knowledge-intensive environment of the professional sector, where partnerships are prevalent.[10] Our results are related to Levin and Tadelis [2005], who rely on partnership's comparative advantage in industries where employee quality is hard to evaluate, and to Morrison and Wilhelm [2004], who focus on partnership's impact on fostering mentorship relations.

2 Model

Two partners, who are risk-neutral and discount the future at a rate $r > 0$, play the the following infinite horizon game. At every moment in time, $t \geq 0$, each partner i chooses effort a_t^i from an interval $[0, A]$.[11] Time t total effort contributes to the stock of fundamentals of the partnership, μ_t, which depreciates over time at a constant rate $\alpha > 0$. At any point in time, stock of fundamentals is the mean of the partnership flow profits dY_t,

$$d\mu_t = (r + \alpha)\,(a_t^1 + a_t^2)dt - \alpha\mu_t dt + \sigma_\mu dB_t^\mu, \tag{1}$$
$$dY_t = \mu_t dt + \sigma_Y dB_t^Y,$$

[9]See, also, Williams [2011], Sannikov [2014], and Prat [2015], who provide analytical conditions on the solution of the relaxed problem, under which the first-order approach is valid.

[10]See Empson [2001] and Broschak [2004] for further references.

[11]The upper bound on effort is used to guarantee boundedness of continuation value in Propositions 2 and 4, part iii). In all simulations, as well as in Theorem 3, the bound A is large enough so that equilibrium efforts are interior. Lemma 3 in Appendix A.4 bounds the relational incentives and, so, the efforts in a near-optimal equilibrium.

where $\{B_t^\mu\}$ and $\{B_t^Y\}$ are two independent Brownian Motions.[12] The multiplicative constant, $r+\alpha$, normalizes the total productivity of effort to one, regardless of the depreciation rate of the fundamentals or of the discount rate.[13] Finally, profits are the only publicly observable signal.

Exerting effort a entails a private flow cost $c(a)$, where $c(\cdot)$ is a twice differentiable, strictly convex cost of effort function. We normalize $c(0)=0$ and $c'(0)=\frac{1}{2}$ (see Proposition 1), and in some of the results we further restrict the cost function to be quadratic (see Section 3.2). Finally, at each point in time, partners split the profits evenly. Thus, for fixed effort choices of both partners, a player's continuation payoffs are given by

$$W_\tau^i = \mathbb{E}_\tau^{\{a_t^1, a_t^2\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{\mu_t}{2} - c(a_t^i)\right)dt\right].$$

The partnership model has three features that go beyond the classic repeated-game framework: effort has persistent effect, state is imperfectly monitored, and partners keep on learning about the fundamentals. Specifically, fundamentals, which are the state variable in the game, change only gradually over time driven by the efforts of the partners. Persistence of fundamentals implies that actions have a persistent effect: total effort today adds to the fundamentals, and also to the profit flow, at any later time,

$$\mu_\tau = e^{-\alpha\tau}\mu_0 + \int_0^\tau e^{-\alpha(\tau-t)}(r+\alpha)\left(a_t^1 + a_t^2\right)dt + \sigma_\mu B_\tau^\mu.$$

Secondly, fundamentals need not be observed by the partners, who observe only noisy profit signals. Together with persistence, this implies that all future profits are useful signals of current efforts (see Proposition 2). Thirdly, fundamentals need not be determined by the efforts, and are changing stochastically. Alternatively, fundamentals are a sum of two terms: one that depends entirely on the past efforts of the partners, and

[12] Unless specified explicitly, all processes in this paper are indexed by $t \geq 0$.

[13] The constant is analogous to $1-\delta$, which scales the stage game payoffs in repeated game analysis, where δ is the discount factor. The only results in which the normalization plays a role are the comparative statics in Propositions 3, in which we show that equilibria with a nontrivial level of effort exist as $r+\alpha+\gamma$ converges to zero—even as the marginal benefit of effort on fundamentals, $r+\alpha$, disappears—and no effort is exerted in any equilibrium as $r+\alpha+\gamma$ converges to infinity—even as the effect of effort on fundamentals gets arbitrarily high.

Intuitively, dropping the normalization would make the results easier. Formally, both results continue to hold without the normalization, as we verify at the end of each proof, in Appendix A.3.

the other that is purely stochastic, and reflects an unknown quality of the partnership. Consequently, in equilibrium partners do not know and keep on learning about the fundamentals, or the quality of the partnership (in the spirit of career concern literature, see Holmström [1999]).

The three features are parametrized in the model by α, $\sigma_Y, \sigma_\mu \geq 0$.[14] Their impact on the incentive provision in a partnership is one of the central themes of the paper, and we discuss it at length in the following sections.

Public Beliefs Let $\overline{\mu}_\tau = \mathbb{E}_\tau^{\{a_t^1, a_t^2\}}[\mu_\tau]$ denote the public expected level of fundamentals at time τ, given efforts $\{a_t^1, a_t^2\}$. A simple application of the Kalman-Bucy filter yields that $\overline{\mu}_t$ follows

$$d\overline{\mu}_t = (r+\alpha)\,(a_t^1 + a_t^2)dt - \alpha\overline{\mu}_t dt + \gamma_t[dY_t - \overline{\mu}_t dt], \tag{2}$$

for an appropriate gain parameter γ_t, $dY_t = \overline{\mu}_t dt + \sigma_Y dB_t$, and a Brownian Motion $\{B_t\}$. We assume that, initially, partners believe that μ_0 is Normally distributed with steady-state variance σ^2 (we relax this assumption in Appendix C.1). This implies that both the posterior estimate variance σ_t^2 and the gain parameter γ_t remain constant throughout the game and equal (see Liptser and Shiryaev [2013])

$$\gamma = \sqrt{\alpha^2 + \frac{\sigma_\mu^2}{\sigma_Y^2}} - \alpha, \text{ and } \sigma^2 = \gamma \times \sigma_Y^2. \tag{3}$$

2.1 Equilibrium

A player's (pure, public) strategy $\{a_t^i\}$ is a process that depends on the public information $\{Y_t\}$ and allows for public randomization. A pair of public strategies, $\{a_t^1, a_t^2\}$, is a *Perfect Public Equilibrium (PPE)* if, for each partner i at any time $\tau \geq 0$,

$$\mathbb{E}_\tau^{\{a_t^i, a_t^{-i}\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{\mu_t}{2} - c(a_t^i)\right)dt\right] \geq \mathbb{E}_\tau^{\{\widetilde{a}_t^i, a_t^{-i}\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{\mu_t}{2} - c(\widetilde{a}_t^i)\right)dt\right], \tag{4}$$

[14] We require that either σ_Y or σ_μ is strictly positive, to avoid the familiar complications in defining a continuous-time strategy in a game with perfect monitoring.

following any history, for any possible alternative strategy $\{\widetilde{a}_t^i\}$.[15] We note that a public strategy does not restrict a partner to condition only on the public expectation $\overline{\mu}_\tau$, or to revert to the equilibrium path immediately after a deviation. Indeed, establishing conditions under which "double deviations" are not optimal is one of the main technical results in the paper (see Theorem 3).

Markov Equilibria In a stochastic game, one often restricts attention to *Markov Equilibria,* in which play depends on the past history only via the minimal set of payoff-relevant parameters[16]. The following result shows that our game has a unique Markov equilibrium, and it exhibits constant effort by the partners.

Proposition 1 *A pair of constant strategies $\{a_t, a_t\}$ in which partners never exert effort, $a_t = 0$, for every $t \geq 0$, constitutes a PPE. It is the unique stationary PPE, and so it is the unique Markov equilibrium.*

The argument is as follows. Exploiting the exponential decay of the fundamentals, for any PPE strategy profile, we may rewrite the continuation payoffs by additively separating the impact that past actions and future actions have on future profits. This gives us

$$\begin{aligned} W_\tau^i &= \mathbb{E}_\tau^{\{a_t^1, a_t^2\}} \left[\int_\tau^\infty e^{-r(t-\tau)} \left(\frac{\mu_\tau}{2} e^{-\alpha(t-\tau)} + \int_\tau^t (\alpha + r) \frac{a_s^1 + a_s^2}{2} e^{-\alpha(t-s)} ds - c(a_t^i) \right) dt \right] \\ &= \frac{1}{2(r+\alpha)} \mathbb{E}_\tau^{\{a_t^1, a_t^2\}} [\mu_\tau] + \mathbb{E}_\tau^{\{a_t^1, a_t^2\}} \left[\int_\tau^\infty e^{-r(t-\tau)} \left(\frac{a_t^1 + a_t^2}{2} - c(a_t^i) \right) dt \right]. \end{aligned} \quad (5)$$

The first term in the last line of (5) captures the expected value of inherited fundamentals to a partner. Even if at some time τ partners stop exerting effort, they will keep collecting partnership profits. The "inherited" (expected) fundamentals $\overline{\mu}_\tau$ may be positive due to past effort or luck—high quality of the partnership—and are expected

[15] In our Gaussian setting a mixed action at any time generates the same effect on the state variable and the same public signal as its expectation. As it is strictly costlier, it is never optimal. Moreover, as players have no private signals in the game, any pure strategy is public.

[16] See Maskin and Tirole [2001] and Mailath et al. [2006] for the formal definition of Markov Equilibrium.

to slowly revert to zero, yielding expected profits all along. The second term is the forward-looking value of efforts undertaken in the future.

Crucially, due to the additive separation of the continuation value in (5), the incentives for effort do not depend on the (expected) level of fundamentals (see Holmström [1999]). Partners do not work to invest in a better production technology, as both the marginal effect of effort on fundamentals, $(r+\alpha)\,dt$, as well as the marginal value of an extra unit of fundamentals, $\frac{1}{2(r+\alpha)}$, are constant. Thus, the effort a_M in the unique Markov equilibrium is constant, with marginal cost one half.

Our assumptions on the cost of effort normalize both the level of effort, as well as the expected future value of the partnership in the Markov equilibrium, to zero. We say that a partnership *unravels* if, from that point on, partners exert no more effort—that is, they play the Markov equilibrium.

We highlight that the Markov equilibrium is inefficient. As partner's effort benefits the two partners equally, the marginal social benefit of effort is twice higher. The socially efficient level of effort, a_{EF}, is constant, with marginal cost one.

Strongly Symmetric Equilibria The provision of relational incentives is hindered by the parsimonious information structure of the partnership game. The only information about the partners' efforts comes from the stream of profits. As both partners' efforts enter profits additively, it is not possible to identify which of the partners did, and which one did not, contribute to the common good (Fudenberg et al. [1994]). Thus, as in the classic analysis of repeated duopoly by Green and Porter [1984] or of partnerships by Radner et al. [1986], it is not possible to provide incentives by "transferring" continuation value between the agents, shifting resources from likely deviators. Moreover, asymmetric play is inefficient since the cost of effort is convex, and it does not affect the signals' informativeness of efforts.

Therefore, we concentrate throughout the paper on equilibria in which players choose symmetric strategies, conditioning the provision of effort on the public history available to them in the same way. Formally, a *Strongly Symmetric Equilibrium (SSE)* is a PPE in which the strategies $\{a_t^1, a_t^2\}$ satisfy $a_\tau^1 \equiv a_\tau^2$, after every public history in $\mathcal{F}_t$.

Accounting of Incentives and Local Strongly Symmetric Equilibria In this section, we define non-Markovian, relational incentives that can bridge part of the gap to the efficient level, and that we analyze in the rest of the paper. Towards that goal, first, define *relational capital* w_τ as the expected discounted payoff from future efforts, or the continuation value net of the expected value of the current fundamentals,

$$w_\tau := W_\tau - \frac{1}{2\,(r+\alpha)}\mathbb{E}_\tau^{\{a_t^1,a_t^2\}}[\mu_\tau] = \mathbb{E}_\tau^{\{a_t,a_t\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(a_t - c(a_t)\right)dt\right]. \tag{6}$$

Relational incentives are constructed by conditioning future play, and so relational capital, on public signals. Specifically, when a partner increases effort, future high profit outcomes become more likely. For fixed strategies of the partners, this changes the probability distribution of efforts in the future. We define *relational incentive* F_τ as the marginal benefit of effort net of Markov incentives, or, equivalently, as the marginal effect of effort on relational capital.

Formally,

$$F_\tau := \frac{\partial}{\partial \varepsilon}\mathbb{E}_\tau^{\{a_t,a_t\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(a_t - c(a_t)\right)dt\right], \tag{7}$$

for revenue processes $dY_t^\varepsilon = \overline{\mu}_t^\varepsilon dt + \sigma_Y dB_t$, where $\overline{\mu}_\tau^\varepsilon = \overline{\mu}_\tau + \varepsilon\,(r+\alpha)$ and $\overline{\mu}_t^\varepsilon$ evolves as in (2), with $\varepsilon > 0$.

A necessary condition for a symmetric equilibrium is that effort is locally optimal. That is, for a level of relational incentives F,

$$a(F) = \arg\max_a \left\{(F + 1/2)\times a - c(a)\right\}. \tag{8}$$

A *local Strongly Symmetric Equilibrium (local SSE)* is a profile of symmetric strategies such that, following any history, actions are locally optimal, $a_\tau = a(F_\tau)$, for the function $a(\cdot)$ defined in (8), and F_τ as in (7).

3 Solution

3.1 Payoffs

We begin to solve our partnership game by characterizing the set of relational capitals, and hence payoffs, that can be achieved by the partners in a local strongly symmetric equilibrium. Towards that goal, the following proposition shows how relational capital and relational incentives must evolve in a local SSE. In this section, fundamentals can be deterministically determined by past efforts or stochastic, $\sigma_\mu \geq 0$, but they are imperfectly monitored, $\sigma_Y > 0$.

Proposition 2 *A symmetric strategy profile $\{a_t, a_t\}$ with relational capital and relational incentives processes $\{w_t\}$ and $\{F_t\}$ is a local SSE if and only if there are L^2 processes $\{I_t\}$, $\{J_t\}$ such that*

$$
\begin{aligned}
dw_t &= \left(rw_t - (a_t - c(a_t))\right) dt + I_t \times (dY_t - \overline{\mu}_t dt) + dM_t^w, \\
dF_t &= (r + \alpha + \gamma) F_t dt - (r + \alpha) I_t dt + J_t \times (dY_t - \overline{\mu}_t dt) + dM_t^F,
\end{aligned}
\tag{9}
$$

and actions satisfy $a_t = a(F_t)$, where $\{M_t^w\}$ and $\{M_t^w\}$ are martingales orthogonal to $\{Y_t\}$, and the transversality conditions $\mathbb{E}\left[e^{-rt}w_t\right], \mathbb{E}\left[e^{-(r+\alpha+\gamma)t}F_t\right] \to_{t\to\infty} 0$ hold.

The first equation in the Proposition is a version of the standard "promise keeping" accounting for the continuation value (see Sannikov [2007]). If the current flow of relational capital is lower than the average promised flow, then the relational capital must deterministically increase in the next period, and vice versa. Moreover, relational capital also changes stochastically in response to the unexpected profit realizations, with linear sensitivity I_t. The martingale processes capture the possibility of public randomization.

The second equation means that relational incentives equal the expected discounted integral of the stream of future sensitivities $(r + \alpha) I_t$ (see, also, Prat and Jovanovic [2014], Sannikov [2014], and Prat [2015]),

$$
F_\tau = \mathbb{E}_\tau^{\{a_t, a_t\}} \left[\int_\tau^\infty e^{-(r+\alpha+\gamma)(t-\tau)} (r + \alpha) I_t dt \right]. \tag{10}
$$

A key intuition behind the equation is that a deviation to a higher effort today results in the partnership's fundamentals above the publicly expected level of fundamentals not only now, but throughout the future. This means unexpectedly good news—profits higher than expected—which keep pushing relational capital up (when sensitivities I_t are positive, see Equation 9).

After a deviating effort, the wedge between the private and public expectation of the fundamentals reverts to zero gradually, at a rate $\alpha + \gamma$. The first term is the exogenous rate of decay of the fundamentals. It is the rate at which the effect of current effect on fundamentals wears off. The second term is the endogenous speed of learning from profits about the fundamentals (see Equation (2)). For instance, an off-equilibrium increase in effort leads to an unexpectedly high stream of profits. Upon observing it, the public attributes part of the higher profits to a permanent change in partnership's quality (due to quality being stochastic) and part of it to transient luck this period (due to imperfect monitoring). The first effect is incorporated into higher expectation of fundamentals, and hence to future expectations of profits. Hence, as the stream of higher-than-expected profits realizes, the wedge between the private and public expectation shrinks.

One way to think about the effect of learning is that, following an off-equilibrium increase in effort, the realized higher-than-expected profits are gradually misattributed to a permanent exogenous change in partnership's quality (in similar fashion as Holmström [1999]). Note also that when fundamentals are deterministic, $\sigma_\mu = 0$, partners do not learn in equilibrium, $\gamma = 0$. We discuss the effect of learning on relational incentives in more detail in Section 4.1.

We now state one of the main results of the paper, characterizing the supremum continuation value in a partnership. Recall that it equals the supremum relational capital, plus the the value of the inherited fundamentals (see Equation 6).

Theorem 1 *Let w^* be the supremum of the relational capitals achievable in a local SSE. The upper boundary of relational incentives achievable in a local SSE is concave and* *satisfies the differential equation*

$$
\begin{aligned}
(r+\alpha+\gamma)F(w) &= \max_{I}\left\{(r+\alpha)I + F'(w)\left(rw - [a(F(w)) - c(a(F(w)))]\right) + \frac{F''(w)\sigma_Y^2}{2}I^2\right\} \\
&= F'(w)\left(rw - [a(F(w)) - c(a(F(w)))]\right) - \frac{(r+\alpha)^2}{2\sigma_Y^2 F''(w)}, \qquad (11)
\end{aligned}
$$

on $[0, w^*)$, *as well as the boundary conditions*

$$
\begin{aligned}
F(0) &= 0, \\
\lim_{w\uparrow w^*}\left\{(r+\alpha+\gamma)F(w) - F'(w)\left(rw - [a(F(w)) - c(a(F(w)))]\right)\right\} &= 0, \qquad (12) \\
\lim_{w\uparrow w}\left\{rw - [a(F(w)) - c(a(F(w)))]\right\} &= 0.
\end{aligned}
$$

Moreover, w^* *is not attained by any local SSE.*

Theorem 1 provides a characterization in form of an HJB equation of the upper boundary of relational incentives achievable in a local SSE. We explain why the result is not an application of standard dynamic programming techniques, and the difficulties involved in the proof, in Section 5. The main value of the result is as a tool to compute the supremum w^* of relational capitals achievable in a local SSE, as the right-most argument of a function that solves this system.

In the first line of Equation (11), the left-hand side is the average flow of relational incentives needed to generate the stock of relational incentives $F(w)$, given the exponential discounting, mean reversion, and learning. On the right-hand side, the first term in brackets captures the flow of relational incentives; the second term captures the change in the relational incentives resulting from the drift in relational capital; and the last term captures the loss (since the boundary is concave) resulting from the second-order variation in relational capital.

The first boundary condition in (12) says that the relational incentives in any local SSE with no relational capital must be zero—just as in the Markov equilibrium.[17] The

[17] The result follows from our assumption that the Markov equilibrium effort 0 is also the lowest available effort. Allowing negative efforts and, thus, efforts below the Markov level, will allow players to "burn more value" in equilibrium and might help enlarge the set of SSE. Formally, with negative effort,

following two equations show that, close to the right end of the boundary, both the drift and the volatility of relational capital die out.[18]

The theorem also shows that the supremum relational capital is not attainable, and so an optimal local SSE does not exist. This follows from the last two boundary conditions in (12), which imply that the supremum relational capital would have to be the outcome of a stationary—and, hence, Markov—equilibrium.

Existence of non-trivial equilibrium and the discounting of incentives Relational incentives are discounted at a rate $r + \alpha + \gamma$. The next proposition establishes that the characterization above is not vacuous, and nontrivial local SSE exist exactly when this "discount rate" is low.[19] In order to simplify the construction of equilibria, for the remaining results we assume that the cost of effort is quadratic:[20]

$$\text{(Quadratic Cost)} \qquad c(a) = \frac{1}{2}a + \frac{C}{2}a^2. \tag{13}$$

Proposition 3 *Fix the ratio $\frac{r+\alpha+\gamma}{r}$. i) The supremum w^* of relational capitals achievable in a local SSE is strictly positive when $r + \alpha + \gamma$ is sufficiently small.*

ii) In contrast, if $r + \alpha + \gamma$ is sufficiently large, then the supremum w^* of relational capitals achievable in local SSE is arbitrarily close to zero.

The "discount rate" $r+\alpha+\gamma$ determines the time horizon for the provision of incentives (see Proposition 2). To motivate today's effort when the rate is high, high profit outcomes must be rewarded soon—either because partners do not care much about the future, the

the differential equation (11) and the right boundary conditions in Theorem 1 would not change, but the left boundary condition $\underline{w} \leq 0$ would become free. (It is easy to establish that just as $(0, 0)$, the point $(\underline{w}, F(\underline{w}))$ must belong to the set of pairs (w, F) with zero drift.)

[18] Positive drift or volatility would lead to an escape beyond the right end. Also, if the drift were strictly negative, one could generate relational capital above w^* simply by letting it drift down.

[19] Formula 24 in Appendix A.3 provides precise sufficient conditions on the parameters that guarantee existence of local SSE; in the proof of Theorem 3 we provide conditions for global incentive compatibility.

[20] Quadratic costs greatly simplify deriving the bounds in Proposition 3, and in Theorems 2 and 3, but we are confident that the result can be extended to more general cost functions, with appropriate bounds on third derivatives.

effect of effort on profits wears off quickly, or the effect is quickly attributed to a change in partnership's quality.

Relational incentives die out when the "discount rate" is high enough for the following two reasons. The first reason is specific to relational incentives: at the bliss point of highest relational capital, partners cannot be rewarded for high profit outcomes. This is because rewards must be meted out in increased relational capital, and this is not available at the bliss point (relational capital is already at the highest). The second reason relies on the "short periods", with signals coming in continuously and hence with small precision. With poor quality of signals, instant incentives require both punishments and rewards—for bad and good signals, respectively (see Abreu et al. [1991] and Sannikov and Skrzypacz [2007, 2010]). Intuitively, with imprecise signals, rewards and punishments are used incorrectly very often, and so both are needed for the two errors to cancel out.[21] It follows that, if the time horizon for incentive provision is short, then the partners cannot be incentivized to exert effort at the bliss point. The construction of relational incentives essentially unravels.[22]

In contrast, with low "discount rate" on incentives, nontrivial relational incentives are possible. In an equilibrium, good profit outcomes are always rewarded, when the relational capital of the partnership is in the workaday interior ranges. Hence, upon reaching the bliss point, partners exert effort because it will be rewarded later, once the relational capital drifts down. Waiting does not destroy much of the incentives since the discounting is low.

3.2 Strategies and Equilibrium Dynamics

In the next result, we construct near-optimal local SSE. To guarantee existence, we restrict attention to a class of local SSE, with sensitivities I_t of relational capital with

[21] In the continuous-time Gaussian setting, the incentives have even more structure: not only both rewards and punishments must be used, but relational capital must be linear in the Gaussian signal; see Proof in Appendix A.3.

[22] Relational capital at a bliss point must be low, because it is a weighted average between the current benefit—close to zero, when efforts are low—and the future relational capital, which can only be lower.

respect to profit flow either zero or above ε, for $\varepsilon > 0$.[23] A local SSE is called ε-*optimal* if it belongs to such class and gives rise to relational capital close to the supremum.[24]

Theorem 2 *For $\varepsilon > 0$, let w^*_ε be the supremum of relational capitals achievable in a local SSE with sensitivities I_t of relational capital with respect to profit flow either zero or above ε. The upper boundary of relational incentives achievable in a local SSE under this constraint is concave and satisfies the differential equation*

$$(r+\alpha+\gamma)F_\varepsilon(w) = \max_{I_\varepsilon \in \{0\}\cup[\varepsilon,\infty)} \left\{ (r+\alpha)I_\varepsilon + F'_\varepsilon(w)\left(rw - [a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))]\right) + \frac{F''_\varepsilon(w)\sigma_Y^2}{2} I_\varepsilon^2 \right\} \quad (14)$$

*on $[0, w^*_\varepsilon)$, as well as the boundary conditions (12). There exists an ε-*optimal *local SSE with relational capital and incentive processes $\{w_t\}$, $\{F_\varepsilon(w_t)\}$.*

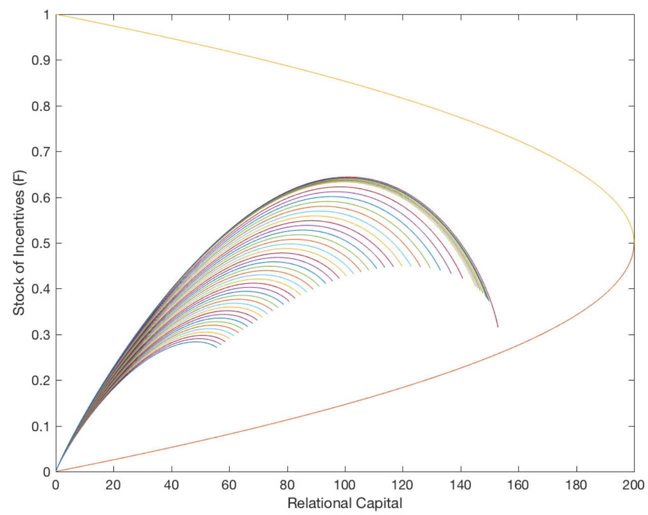
The function F_ε in the theorem provides a recipe for constructing near-optimal local SSE. In Figure 1, F_ε is the highest inverse parabola, which reaches furthest to the right. At any point in time, for any value of relational capital (state), the function determines relational incentives, and so the marginal benefit of effort. This pins down the equilibrium effort and also the relational capital in the next instant: it drifts deterministically—say, decreases if the flow benefits are large relative to the relational capital—but also responds to the stochastic news about the profit flows (see (2)). The sensitivity to those news is the one that maximizes expression (14) and, again, is pinned down by function F_ε and its second derivative. In the next instant, the game continues with updated relational capital (as described) and beliefs about the fundamentals (see Equation 2).

We point out three qualitative features of the equilibrium dynamics. First, on the left,

[23] Specifically, we are looking at a subclass of the local SSE in the *original* game that happen to have this property. The advantage of local SSE in this class is that, for a fixed $\varepsilon > 0$, they cannot run into trouble approximating the unattainable w^*, with policies $\{I_t\}$ positive yet arbitrarily small. This yields self-generation of the upper boundary F_ε in the following theorem.

We point out that there might be other types of approximately optimal local SSE. The class that we chose has an additional benefit of numerical tractability; in Appendix A.4 we provide the bounds on the first two derivatives of the functions F_ε in Theorem 2. In contrast, note that the equation (11) in Theorem 1 is not uniformly elliptic, with F'' arbitrarily small.

[24] Formally, we require the distance to the supremum to be vanishing in ε. The equilibria in the next theorem, in particular, achieves distance of order $O(\varepsilon^{1/3})$.



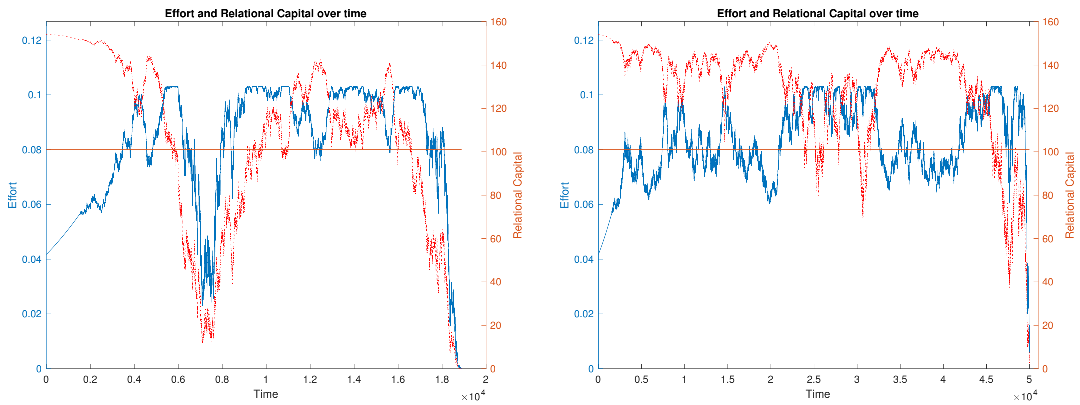
This figure displays many different solutions of the differential equation (14), with the near-optimal local SSE characterized by the curve that reaches farthest to the right. The horizontal parabola is the locus of the feasible relational capital-incentives pairs (w, F) that can be achieved by symmetric play in a stage game, satisfying $rw = a(F) - c(a(F))$. The efficient pair is $(200, 1/2)$.

Figure 1: Relational Incentives in a Near-optimal SSE

the graph of F_ε starts at the Markov equilibrium point, with no relational capital and incentives. This is an absorbing, stationary point; if partners reach it after a sequence of bad profit outcomes, the relationship unravels, and no effort is ever taken in the future. Second, on the right, close to the bliss point, profit outcomes hardly affect the partnership. Once partners have established a sufficiently high level of relational capital, it becomes relatively insensitive to the public news and slowly drifts down, away from the boundary. Third, profit outcomes that exceed expectations are always good news for the partnership, increasing relational capital. However, they do not always lead to greater effort. Function F_ε is concave and changes slope, and so good news increase incentives at low levels, and decrease incentives at high levels of relational capital. This translates into the equilibrium dynamics of effort:

Corollary 1 *In a near-optimal local SSE there is a threshold level of relational capital, $w^\#$, such that i) at relational capitals below $w^\#$ high profit realizations dY_t increase equi-*

librium effort ("rallying"), and ii) at relational capitals above $w^{\#}$ high profit realizations dY_t decrease equilibrium effort ("coasting").



(a) A short-lived partnership (b) A long-lived partnership

Each panel displays a sample path of effort (on the left axis) and relational capital (on the right axis) over time, starting near the supremum relational capital. The horizontal line represents the level of relational capital at which effort is maximized. Initially players coast, and the relational capital drifts down, undisturbed by shocks. When relational capital is above the horizontal line, profit outcomes that increase relational capital lead players to exert less effort. Changes in effort and relational capital are negatively correlated. When relational capital is below the horizontal line, changes in effort and relational capital are positively correlated.

Figure 2: Effort and Relational Capital over Time

4 Information Structure and the Value of the Partnership

The informational environment of a partnership, in our setting, is determined by two parameters. First, partnerships differ by how well their fundamentals are monitored, and so how well the progress of the venture can be tracked and assessed. This is captured by the degree of noise in the public signals, σ_Y. Second, partnerships differ by the degree of underlying uncertainty about the quality of the venture, or the partners. This is captured by the degree of volatility of the fundamentals, σ_μ. We investigate those two dimensions and their effect on partnership's value in turn.

4.1 Monitoring the Partnership

The equilibrium in our dynamic environment is inefficient because partners do not observe each other's effort. Otherwise, they could sustain efficiency by reverting to the inefficient Markov equilibrium following any spat of shirking.[25] It seems thus compelling that better monitoring of the fundamentals in our game should increase the partnership's value. The following proposition shows that this intuition captures only part of the story.

Proposition 4 *i) Suppose fundamentals are deterministic, $\sigma_\mu = 0$. The supremum w^* of relational capital achievable in a local SSE is increasing in the precision of the monitoring technology σ_Y^{-1}.*

ii) Suppose the fundamentals are stochastic, $\sigma_\mu > 0$. The supremum w^* of relational capital achievable in a local SSE is arbitrarily close to zero, when monitoring is precise enough (σ_Y^{-1} sufficiently large).

iii) Suppose the fundamentals are stochastic, $\sigma_\mu > 0$, and monitored perfectly, $\sigma_Y = 0$. The unique SSE is the Markov equilibrium, with relational capital $w = 0$.

When there is no uncertainty about the quality of the partnership and fundamentals are deterministic, better monitoring always improves efficiency (part (i) of the proposition). The intuition is simple: absent uncertainty about the quality, the public signals are used solely as signals of effort, and better monitoring mitigates informational frictions.[26]

When the quality of the fundamentals is uncertain, partners use public signals not only to incentivize effort, but also to estimate the quality of the partnership. Better monitoring still benefits the partnership by providing better signals of effort,[27] but it also

[25]Modeling a continuous-time game with perfect monitoring runs into the usual problems. However, "Grim-Trigger" strategies approximate efficiency in a discrete-time approximation of the game, given that periods are "short" and so discount factor arbitrarily close to one, and the MinMax strategy is a stage-game Nash equilibrium.

[26]Note that providing the same level of incentives with less noise requires less variability of relational capital in equilibrium. Suppose σ_μ, $r = 1$, $\alpha = 0$; generating relational incentives F of, say, one, requires sensitivity I of relational capital to public signal equal one as well. This results in the volatility of relational capital σ_Y, increasing in noise. Formally, the only impact of σ_Y on the HJB equation (11) is through the last term, with the cost of incentives due to the second-order variation of relational capital increasing in σ_Y.

[27]See also the impact of improved monitoring on the incentivizing wage, in Appendix B. The impact of better monitoring on the variability of relational incentives, as in the previous footnote, is analogous.

results in better learning about the fundamentals (higher gain parameter γ). Crucially, faster learning means that good outcomes are quickly incorporated in increased expected fundamentals, and the window for rewarding unexpectedly high outcomes, and so effort, shrinks. This shorter horizon for the incentive provision is particularly harmful at the bliss point of the partnership, when effort can be motivated solely by future rewards (Proposition 3). Part (ii) of Proposition 4 establishes that, with little noise, this negative effect is dominant and eliminates relational incentives.[28]

In the extreme, if fundamentals are monitored with no noise, the current change in fundamentals is a sufficient statistic to evaluate current effort. Incentives must be provided immediately, as in the repeated game i.i.d. setting. Since this is not possible at the bliss point of maximal relational capital, the construction of any relational incentives unravels (part (iii) of the proposition). This impossibility is directly related to the results in Sannikov and Skrzypacz [2007, 2010].

One solution to this impossibility, proposed by Abreu et al. [1991], is to withhold the arrival of information; players observe the relevant path of signals only at times $l, 2l, 3l$, etc., for a fixed time length $l > 0$. In the new game—with "compounded" periods, actions, and signals—the horizon for incentive provision is still only the current period. However, the bundling of information improves the information quality in any given period and partners can be incentivized to work even when the relationship is at its best.

In this paper, we highlight an alternative solution, and the benefits of *poorer* monitoring of the state. Our results show that with perfectly monitored fundamentals it is not the "short periods", but the instantaneous flow of information and time horizon for incentives that hampers the provision of incentives in the partnership.

Structuring Joint Economic Activity. Throughout the paper we have focused on joint enterprises that are organized as partnerships, but modern economies display

[28]Note that, as σ_Y shrinks, only the left-hand side (required mean flow of incentives) and the last term in the HJB equation 11 (contribution of the incentive flow) are scaled up. When the middle term capturing the benefit of delayed incentives disappears, the effect is similar as when the horizon for incentive provision shrinks. Formally, the solution of the HJB equation that starts around $w^* > 0$ would reach arbitrarily high levels, since i) $F''(w)$ is bounded away from negative infinity, as long as $F(w)$ is bounded away from zero; and ii) $F'(w)$ is arbitrarily steep close to w^* (Theorem 1).

a large variety of organizational structures. The optimal provision of incentives must be tailored to the informational constraints and to the organizational structure of the enterprise. For instance, in partnerships, the common ownership structure fosters long-term relationships, hence relational incentives are key. In contrast, in other organizational structures—such as coorporations—the principal or owner of the enterprise relies on contracts and performance-based wages, or career concerns to incentivize each of its employees.[29]

The mechanism that underlies the benefit of poor monitoring in an ongoing partnership, explained above, is peculiar to the provision of relational incentives. It relies on two key elements. First, poor monitoring extends the time horizon for incentive provision; and, second, partnerships are not able to provide immediate relational rewards at the bliss point—when relational capital is maximal. We argue below that this mechanism is absent and, as a result, poor monitoring has always adverse effect on incentives derived from career concerns and performance-based wages. These contrasting comparative static results have implications on the structuring of a firm across informational environments.

Building own reputation provides an alternative mechanism to incentivize effort (*career concerns*, Holmström [1999]). Just as in the case of the long-term relational incentives, reputation building is instrumental in very weak contractual environments, when the publicly observable outcomes are not contractible, or payments schemes that condition on it are not enforceable. In Appendix B we provide a version of Holmstrom's career concern model adapted to our setting, with two workers, the same production technology—with fundamentals driven by two efforts and two qualities of the workers—and the same information structure.

With career concerns, a worker exerts effort so that future good profit outcomes are misattributed to his high quality, reflected in a higher competitive wage. When public news becomes more informative about the fundamentals of the enterprise, good outcomes increase the expected fundamentals of each worker faster, as with partnership. However, differently than in a partnership, this has unambiguously positive effect on the

[29]See Prendergast [1999] for a review on the broad question of organizational design and its implications on the provision of incentives.

incentives, as the reputational benefits of effort accrue earlier on (see also Holmström [1999] Proposition 1).[30] Hence, improved monitoring and the shortening of the horizon of incentives facilitates the provision of career-concern's incentives.

Allowing worker's compensation to be conditioned on public signal opens up a much wider scope of *contracts* and performance-based compensation. These schemes rely on detailed information being contractable and the heavy demands (in terms of both contractability and enforceability) of such schemes has been widely discussed in the career concerns and relational contracting literature. Rather than ignoring contracts entirely, we introduce a simple model with costly (stationary and linear) transfers—wage payments—that condition on public news.[31] We provide further details in Appendix B.

The cost of incentivizing effort using performance-based wages is a consequence of the information asymmetry in the environment. Improved monitoring improves public signals' informativeness, alleviating the informational frictions. Formally, we show in Appendix B that with better signals the variance of incentivizing wages, and hence the cost of contracts goes down. Moreover, while we do not discuss this formally, we point out that better monitoring and shorter time lag before the effects of worker's actions materialize puts less commitment demands on contracts. The difficulties associated with "short-terminism" and preventing manipulations that materialize long after the job termination are well know in the contracting literature (see, e.g., Edmans et al. [2012]).

The first prediction that follows from these comparative statics is a cross-section of organizational structures across different industries. Partnerships may benefit when progress of the venture is based on long-term, qualitative contributions. In contrast, corporations, with their reliance on market-based incentives—employees hired for specific tasks and rewarded either by improved reputation or, more directly, by a performance-based wage—benefit when the progress is easier to quantify and measure, or if the data becomes cheaper and more abundant. Hence, partnerships should be more prevalent in industries in which the information about the venture is murkier.

[30] Career concerns benefits also exist at every state of the venture, with no "unravelling at the bliss point".

[31] See, e.g., Tirole [1988] for a textbook treatment of costly transfers.

This prediction is consistent with the fact that partnerships are very common in the professional sector, i.e. law firms, accounting, advertising (see Levin and Tadelis [2005] and Von Nordenflycht [2010]). A key feature of these knowledge-intensive environments is that the quality of a firm's output is "opaque"[32]. Even after the output is produced and delivered, its quality is hard to evaluate. For instance, for an advertising agency, even after the campaign is published its quality and effects are hard to measure: Was the advertising agency's campaign responsible for the sales increase? A similar argument holds for other professional partnerships, i.e. was the lawyer's argument responsible for the acquittal?

A second prediction that follows from these comparative statics is that, within a particular industry, if technological changes lead to improvements in the monitoring of the joint-venture's fundamentals, with feedback about it becoming more abundant, then partnerships should become less common in that industry. This is in line with the trend in the last decades of professional partnerships moving away from fixed surplus sharing rules and into individual productivity-based profit sharing (see Levin and Tadelis [2005],Empson and Chapman [2006], and Empson [2010]).

4.2 Uncertainty about the quality of the partnership

In our environment, there is a second obstacle to partners' monitoring of each other's effort: the quality of the partnership is stochastic and unobserved by the partners. The uncertainty is captured by the degree of volatility of the fundamentals, σ_μ.[33]

In contrast to improved monitoring, reducing uncertainty about the quality of the joint venture facilitates the provision of incentives. This is because reduced uncertainty results in the public news more closely tracking the effort exerted by the partners, instead of reflecting the exogenous changes in quality. We capture this intuition in the following result:

[32] See Empson [2001], Greenwood and Empson [2003], Broschak [2004], and Von Nordenflycht [2010].

[33] More precisely, variance in beliefs is strictly increasing in σ_μ, end equals zero when σ_μ does; see Equation 3.

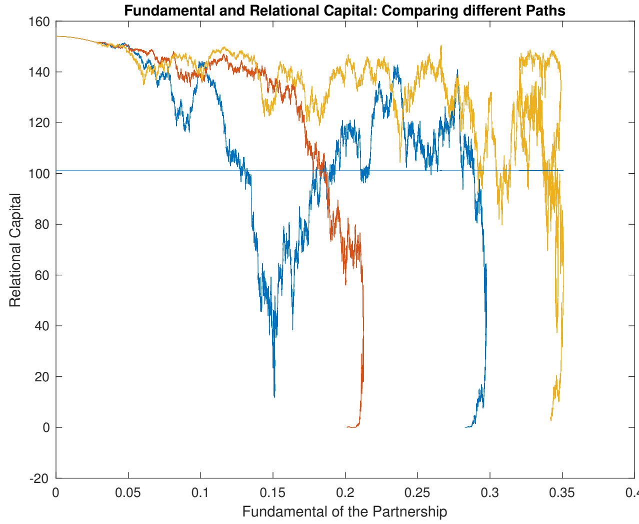
Proposition 5 *The supremum of relational capital, w^*, that the partnership can generate in a local SSE increases with the precision regarding the partnership quality, σ_μ.*

Depending on the level of uncertainty about the venture, a stream of bad outcomes is interpreted differently. For instance, with little uncertainty (σ_μ is low or absent) expected fundamentals barely respond to profit outcomes (as γ is close to zero) and, thus, are much more sluggish than relational capital. In this case, a short string of sharp, low profit realizations will unravel the partnership (Theorem 2), with hardly any effect on its profitability (see Equations (2) and (3)). In other words, even a very profitable partnership may unravel when its goodwill is tested by a series of adverse outcomes, even though these have a negligible effect on the partnership's profitability.

Corollary 2 *In a near-optimal local SSE, at any point in time t, a partnership may unravel in an arbitrarily short period of time after a sequence of unexpected bad news. The accompanying change in the expected profitability is of order σ_μ times the amount of bad news.*

Figure 3 displays the differences in the dynamics of the fundamentals and of the relational capital. It shows three different sample paths, highlighting that a partnership's relational capital is not determined by its profitability. Furthermore, even at dissolution, partnerships have different levels of productivity.

Young and Old Partnerships Partnerships differ in levels of uncertainty about the quality of the venture. In real world environments, a driving force behind such uncertainty is the lack of experience among the partners. Younger joint ventures, tend to have less experience, and hence more uncertainty about their technology, the product or service, the environment, or themselves; while, in mature enterprises, the quality of the joint venture is better known. For tractability, we consider only stationary models, with uncertainty constant over time. We interpret it to be low for mature ventures, and high for the young ones. An alternative would be to consider a fully non-stationary model, with the uncertainty gradually shrinking (See Appendix C.1)



This figure displays three different sample paths of the relational capital of a partnership, as a function of the fundamentals of the relationship. The horizontal line marks the level of relational capital at which effort is maximized.

Figure 3: Relational Capital and Fundamentals of a Partnership

Our results indicate that the use of relational incentives to motivate the partners is more adequate at mature, long-standing relationships. As the partnership is better understood, the partners can use the public news to more precisely reward the provision of effort. In contrast, in young enterprises, the uncertainty about the quality of the joint venture gets confounded with the uncertainty about the level of fundamentals. Hence, if one of the partners free rides, part of the bad news will be attributed to a "worse than expected" quality, inhibiting punishments.

We have discussed in the previous section how career concern incentives, in sharp contrast with our results, are enhanced when the learning about the quality of the venture improves. More uncertainty, and greater scope of learning has precisely the same effect (via greater γ; see Holmström [1999]). More broadly, the scope of incentives that derive from learning and manipulating beliefs of partners or of the market may be large for young partnerships (see, e.g., Bolton and Harris [1999], Bar-Isaac [2007], Bonatti and Hörner [2011]). However, the provision of incentives in mature ones must rely on relational incentives, with the mechanism detailed in this paper.

The proposition provides a novel rationale for why an employment period is common before someone is made a partner in a professional partnership (see Levin and Tadelis [2005], Ghosh and Waldman [2010], and Von Nordenflycht [2010]). While the literature has focused on the partners learning the level of productivity of the candidate and only promoting the candidate if productivity is above a set bar, our result highlights that reducing the magnitude of the uncertainty might play a role as well.

Finally, we shine a light on the puzzling feature of why some partnerships decide to break up, even if highly profitable. For instance, The Beatles (10 years together) and Daft Punk (28 years) in music; Jamie Dimon and Sandy Weill from Citigroup (15 years) in finance; and Daniel Humm and Will Guidara from Eleven Madison Park (13 years) in fine-dining. Our result shows that, for experienced, mature enterprises, a string of bad news can severely deteriorate the relationship and even lead to the dissolution of the partnership, however with minimal effects over the overall profitability of the joint venture. Younger enterprises, with more uncertainty about the partnership quality, tend to burn their perceived productivity before dissolution.

5 Method and Extensions

In the first part of this section, we present the main steps in the proofs of Theorems 1 and 2 (details of the proofs are in Appendix A). We aim to highlight the broader methodological contribution of our results and to explain why we cannot rely on the existing solution methods. An abstract general model, the generalizations of Propositions 6 and 7, and a range of special cases are presented in Appendix C.

In the second part of this section, we provide conditions on the primitives when our "first-order approach" is valid, and a near-optimal local SSE is a SSE.

5.1 Method and proof sketch

Our results rely on a novel parametrization and a corresponding extension of the stochastic control methods. Our method of solving for near-optimal local SSE consists of two steps. First, we parametrize the supremum of relational incentives as a function of relational capital; and, second, we establish that the boundary satisfies the HJB differential equation (11). To the best of our knowledge, the approach of maximizing incentives as a way to characterize the optimal equilibrium, as well as the HJB characterization of the value function, when the value level affects the state variables, are novel.

Let us first discuss why a novel parametrization is needed. One alternative is to consider equilibria in which play, and incentives in particular, depends on the past only via a publicly observable exogenous variable, such as public beliefs $\overline{\mu}_t$ ("Markov in $\overline{\mu}_t$"). By ignoring relational incentives, and with the movement of beliefs exogenously given, the benefit of this approach is its relative simplicity, which is particularly valuable in more complex, nonlinear environments (see Cisternas [2017] for such a characterization of the stock of incentives in a career concerns settings). The downside is that a characterized equilibrium is not optimal.

Another alternative is to characterize the supremum continuation value (in our case, relational capital) using incentives as a state variable, as is customary in Principal-Agent problems, or Markov equilibria.[34] Relative to our solution method, this approach would swap the state variable and the objective, switching the axes in Figure 1. Under this parametrization, however, the upper boundary of supremum relational capital would not be self-generating, with the volatility of the value (relational capital) strictly positive when the argument (local SSE incentives) is at its highest.[35]

Our parametrization avoids those complications. More broadly, our approach allows for flexibility in choosing a parametrization of the boundary of local SSE relational capital-incentive pairs (w, F), as in Sannikov's [2007] "geometric"solution method.

The new parametrization requires, in turn, an extension of the stochastic control

[34]The use of an additional *state variable* that relates to marginal incentives is well established; see e.g. Werning [2001] or Kapička [2013], who introduce expected marginal utility of consumption, Williams [2011], Prat and Jovanovic [2014], Sannikov [2014], and Prat [2015] who introduce expected marginal utility of a state, or marginal incentives, in models of contracting with persistence.

[35]Given the characterization of the stock of incentives F_t in Proposition 2, zero volatility I_t requires positive drift of F_t, contradicting its maximality. Positive volatility means that, in near-optimal local SSE, the pair of relational capital and incentives may pass the extreme point and continue along the *lower* boundary of the set of relational capital-incentive pairs achievable in a local SSE.

methods. This is because it results in the law of motion of relational capital (state variable) that depends—through the choice of effort, $a\left(F\left(w\right)\right)$—on the level of the value function F (see (9)). Theorem 1 establishes that the HJB characterization of supremum incentives is, indeed, valid in this setting. We note that our contribution to the stochastic control theory is related to Sannikov [2007] (see also Faingold and Sannikov [2020]): there, methods are extended to settings, where state variables are affected by the *derivative* of the value function. This is necessary to analyze continuous-time games, with no persistence.[36] In our paper extends the techniques to settings, where state variables are affected by the *level* of the value function. We are convinced that this is an important step for the analysis of continuous-time games with persistence.

The following two propositions are the key methodological results underpinning Theorem 1.[37] The first proposition below shows, intuitively, that the solution to the HJB equation in Theorem 1 provides a lower bound for the supremum of relational incentives achievable in a local SSE.

Proposition 6 *Consider a continuous $I:[\underline{w},\overline{w}]\to\mathbb{R}_+$ and a C^2 strictly concave function $F:[\underline{w},\overline{w}]\to R$ that satisfy the differential equation*

$$(r+\alpha+\gamma)F(w)=(r+\alpha)I(w)+F'(w)\left(rw-[a(F(w))-c(a(F(w)))]\right)+\frac{F''(w)}{2}\sigma_Y^2 I^2(w),\quad (15)$$

where a is defined in (8), such that each boundary point $w^\partial\in\{\underline{w},\overline{w}\}$ together with $F\left(w^\partial\right)$ is either achievable by a local SSE or satisfies

$$\begin{aligned}(r+\alpha+\gamma)F(w^\partial)&=F'(w^\partial)\left(rw^\partial-\left[a(F(w^\partial))-c(a(F(w^\partial)))\right]\right),\\ sgn\left(\frac{\overline{w}+\underline{w}}{2}-w^\partial\right)&=sgn\left(rw^\partial-\left[a(F(w^\partial))-c(a(F(w^\partial)))\right]\right).\end{aligned}\quad (16)$$

Then, for every $w_0\in[\underline{w},\overline{w}]$, there is a local SSE $\{a_t,a_t\}$ achieving $(w_0,F(w_0))$.

In particular, the HJB equation (15) together with the function I that pointwise

[36] Note that the extension is not necessary for the analysis of the Principal-Agent problems, in which Principal faces no incentive problems.

[37] Rest of the proof in the Appendix establishes regularity of the boundary, and the boundary conditions (12).

maximizes the right-hand side gives rise to the HJB equation (11) from Theorem 1. Ito's formula implies that if w_t is the relational capital that follows (9) and F is a solution to (15), then the process $\{F(w_t)\}_{t\geq 0}$ satisfies the differential equation (9) in Proposition 2 with sensitivities $J_t = F'(w_t) \times I(w_t)$. Thus, $\{F(w_t)\}_{t\geq 0}$ captures the associated relational incentives.

When the boundary point is a level of relational capital known to be achievable by a local SSE, upon reaching this point, the game simply follows this local SSE. Under the alternative boundary conditions (16), relational capital is reflected back, and the construction as above continues. [38]

The second proposition below shows, roughly, that the solution to the HJB equation in Theorem 1 provides (locally) the upper bound to the supremum of relational incentives achievable in a local SSE. More precisely, for "slack" $\lambda > 0$, consider a differential equation related to (11),

$$(r + \alpha + \gamma)\tilde{F^\lambda}(w) = \tilde{F^\lambda}'(w)\left(rw - \left[a(\tilde{F^\lambda}(w)) - c(a(\tilde{F^\lambda}(w)))\right]\right) - \frac{(r+\alpha)^2}{2\sigma_Y^2 \tilde{F^\lambda}''(w)} + \lambda. \tag{17}$$

Proposition 7 *For every $\lambda > 0$, there is $\delta > 0$ such no concave solution $\tilde{F^\lambda}$ of the differential equation (17) on an interval $[\underline{w}, \overline{w}]$, with $\left|\tilde{F^\lambda}'\right| \leq 1/\lambda$, satisfies both of the following conditions:*

$$i)\ \tilde{F^\lambda}(\underline{w}) = E(\underline{w}) \text{ and } \tilde{F^\lambda}(\overline{w}) = E(\overline{w}),$$
$$ii)\ 0 < E(w) - \tilde{F^\lambda}(w) \leq \delta, \text{ for } w \in (\underline{w}, \overline{w}),$$

where E parametrizes the upper boundary of relational incentives achievable in local SSE.

The result provides a novel escape argument for our setting. If the law of motion of w_t did not depend on the level of the value, as in a standard stochastic control problem, the result would be true with $\lambda = 0$ and $\delta = \infty$. Any value above the solution of the HJB

[38] More precisely, the stock of incentives $F\left(w^\partial\right)$ at the boundary can be generated by having either $I\left(w^\partial\right) = 0$, with relational capital drifting back inside of $[\underline{w}, \overline{w}]$, or $I(w^\partial) = -2\frac{r+\alpha}{\sigma_Y^2 F''(w)} > 0$. In the proof in Appendix A.4, we show how to reduce the second case to the first one by extending the functions F and I beyond $[\underline{w}, \overline{w}]$, with $I = 0$.

equation could be justified only by it drifting ever higher or along the normal vector (see, e.g., Lemma 4 in Sannikov [2007]). In our setting, value affects the law of motion of the state variable. Thus, relational incentives higher than the solution to (11) may increase the right-hand side of the HJB equation. As we show, when the difference is sufficiently small, the benefits do not outweigh the slack λ in the definition of $\tilde{F}^{\lambda}$ and can only be justified by the value drifting even higher, out of the neighborhood of $\tilde{F}^{\lambda}$.

5.2 Global Incentive Compatibility

So far, we have characterized local equilibria. The following result shows conditions on the primitives, under which local SSE satisfy full incentive-compatibility constraints.[39] Existence of nontrivial SSE follows then from Proposition 3.

Theorem 3 *Fix $\varepsilon > 0$ and consider an $\varepsilon-$optimal local SSE $\{a_t, a_t\}$. Then, $\{a_t, a_t\}$ is an SSE when $C\sigma_Y$ is sufficiently high, where C is the second derivative of the cost function and σ_Y is observational noise.*

In particular, for a fixed ratio $\frac{r+\alpha+\gamma}{r}$, nontrivial SSE exist when $C\sigma_Y$ is sufficiently high and $r+\alpha+\gamma$ sufficiently small.

The problem in establishing global incentive compatibility consists in showing that, after any history, the effort choice is concave. Given that the effort cost function is strictly convex, with second derivative C, this boils down to establishing bounds on how convex the expected benefit of effort is. Crucially, in a dynamic environment with persistence, like ours, a deviation affects the strength of incentives that the agent faces in the future. This knock-on effect makes accounting for the benefits of deviations much more involved than in a static setting, or without persistence.

Following up on this intuition, in order to bound how convex the benefit of effort is, it is sufficient to establish a uniform bound on how sensitive the relational incentives are with respect to public signals. The first part of the proof is related to the results in the

[39]Equation 46 in Appendix A.5 provides a precise sufficient condition on the parameters that guarantees global incentive compatibility.

literature and shows that there are no global deviations from a local SSE if this sensitivity of relational incentives is uniformly bounded (see Williams [2011], Edmans et al. [2012], Sannikov [2014], and Cisternas [2017]).

In the second part of the proof, we bound this endogenous sensitivity of relational incentives by a function of the primitives of the model. This part of the proof relies heavily on the analytical tractability of our solution and, in particular, the bounds on derivatives F'_ε and F''_ε, established in the proof of Theorem 2. Intuitively, large noise σ_Y makes incentives costly, resulting in their low sensitivity and, thus, in their relatively linear benefit of effort.

6 Concluding Remarks

In this paper, we present a dynamic model of partnership whose three central features are effort that shapes a persistent state, imperfect state monitoring, and learning about the state. We develop a method that allows us to characterize near-optimal strongly symmetric equilibria of the game with a simple HJB equation. Its solution describes the supremum of relational incentives achievable in an SSE for a given level of relational capital and fully characterizes equilibrium dynamics in near-optimal equilibria. Imperfect state monitoring extends the time horizon for incentive provision, which helps sustain non-trivial relational incentives and equilibrium effort, helping partners. The model provides a novel rationale for the prevalence of partnerships in environments, in which progress, or product quality are hard to measure. We believe the method will contribute to the analysis of continuous-time games with persistence.

References

Dilip Abreu, David Pearce, and Ennio Stacchetti. Optimal cartel equilibria with imperfect monitoring. *Journal of Economic Theory*, 39(1):251–269, 1986.

Dilip Abreu, David Pearce, and Ennio Stacchetti. Toward a theory of discounted repeated games with imperfect monitoring. *Econometrica*, 58(5):1041–1063, 1990.

Dilip Abreu, Paul Milgrom, and David Pearce. Information and timing in repeated partnerships. *Econometrica: Journal of the Econometric Society*, 59(6):1713–1733, 1991.

Armen A Alchian and Harold Demsetz. Production, information costs, and economic organization. *The American Economic Review*, 62(5):777–795, 1972.

Heski Bar-Isaac. Something to prove: reputation in teams. *The RAND Journal of Economics*, 38(2):495–511, 2007.

Venkataraman Bhaskar. The ratchet effect re-examined: A learning perspective. *Unpublished working paper, University of Texas*, 2014.

Simon Board and Moritz Meyer-ter Vehn. Reputation for quality. *Econometrica*, 81(6): 2381–2462, 2013.

J Aislinn Bohren. Using persistence to generate incentives in a dynamic moral hazard problem. *Unpublished manuscript, University of Pennsylvania*, 2018.

Patrick Bolton and Christopher Harris. Strategic experimentation. *Econometrica*, 67(2): 349–374, 1999.

Alessandro Bonatti and Johannes Hörner. Collaborating. *American Economic Review*, 101(2):632–63, 2011.

Joseph P Broschak. Managers' mobility and market interface: The effect of managers' career mobility on the dissolution of market ties. *Administrative Science Quarterly*, 49 (4):608–640, 2004.

Doruk Cetemen, Ilwoo Hwang, and Ayça Kaya. Uncertainty-driven cooperation. *Unpublished working paper*, 2017.

Gonzalo Cisternas. Two-sided learning and the ratchet principle. *The Review of Economic Studies*, 85(1):307–351, 2017.

Jean-Paul Décamps and Thomas Mariotti. Investment timing and learning externalities. *Journal of Economic Theory*, 118(1):80–102, 2004.

Peter M DeMarzo and Yuliy Sannikov. Learning, termination, and payout policy in dynamic incentive contracts. *The Review of Economic Studies*, 84(1):182–236, 2016.

Alex Edmans, Xavier Gabaix, Tomasz Sadzik, and Yuliy Sannikov. Dynamic ceo compensation. *The Journal of Finance*, 67(5):1603–1647, 2012.

Laura Empson. Fear of exploitation and fear of contamination: Impediments to knowledge transfer in mergers between professional service firms. *Human relations*, 54(7):839–862, 2001.

Laura Empson. Evolution and revolution in the governance of law firms. 2010.

Laura Empson and Chris Chapman. Partnership versus corporation: Implications of alternative forms of governance in professional service firms. In *Professional service firms*. Emerald Group Publishing Limited, 2006.

Eduardo Faingold and Yuliy Sannikov. Reputation in continuous-time games. *Econometrica*, 79(3):773–876, 2011.

Eduardo Faingold and Yuliy Sannikov. Moral hazard in stochastic differential games: Beyond markov equilibrium. *Unpublished working paper*, 2020.

Drew Fudenberg, David Levine, and Eric Maskin. The folk theorem with imperfect public information1. *Econometrica*, 62(5):997–1039, 1994.

George Georgiadis. Projects and team dynamics. *The Review of Economic Studies*, 82 (1):187–218, 2014.

Suman Ghosh and Michael Waldman. Standard promotion practices versus up-or-out contracts. *The RAND Journal of Economics*, 41(2):301–325, 2010.

Edward J Green and Robert H Porter. Noncooperative collusion under imperfect price information. *Econometrica*, 52(1):87–100, 1984.

Royston Greenwood and Laura Empson. The professional partnership: Relic or exemplary form of governance? *Organization studies*, 24(6):909–933, 2003.

Zhiguo He, Bin Wei, Jianfeng Yu, and Feng Gao. Optimal long-term contracting with learning. *The Review of Financial Studies*, 30(6):2006–2065, 2017.

Bengt Holmstrom. Moral hazard in teams. *The Bell Journal of Economics*, 13(2):324–340, 1982.

Bengt Holmström. Managerial incentive problems: A dynamic perspective. *The Review of Economic Studies*, 66(1):169–182, 1999.

Hugo A Hopenhayn and Francesco Squintani. Preemption games with private information. *The Review of Economic Studies*, 78(2):667–692, 2011.

Arantxa Jarque. Repeated moral hazard with effort persistence. *Journal of Economic Theory*, 145(6):2412–2423, 2010.

Marek Kapička. Efficient allocations in dynamic private information economies with persistent shocks: A first-order approach. *Review of Economic Studies*, 80(3):1027–1054, 2013.

Ioannis Karatzas. *Brownian motion and stochastic calculus*, volume 113. Springer, 1991.

Godfrey Keller and Sven Rady. Strategic experimentation with poisson bandits. *Theoretical Economics*, 5(2):275–311, 2010.

Godfrey Keller, Sven Rady, and Martin Cripps. Strategic experimentation with exponential bandits. *Econometrica*, 73(1):39–68, 2005.

Nicolas Klein and Sven Rady. Negatively correlated bandits. *The Review of Economic Studies*, 78(2):693–732, 2011.

Patrick Legros and Steven A Matthews. Efficient and nearly-efficient partnerships. *The Review of Economic Studies*, 60(3):599–611, 1993.

Jonathan Levin and Steven Tadelis. Profit sharing and the role of professional partnerships. *The Quarterly Journal of Economics*, 120(1):131–171, 2005.

Robert S Liptser and Albert N Shiryaev. *Statistics of random processes: I. General theory*, volume 5. Springer Science & Business Media, 2013.

George J Mailath, J George, Larry Samuelson, et al. *Repeated games and reputations: long-run relationships*. Oxford university press, 2006.

Eric Maskin and Jean Tirole. Markov perfect equilibrium: I. observable actions. *Journal of Economic Theory*, 100(2):191–219, 2001.

Alan D Morrison and William J Wilhelm. Partnership firms, reputation, and human capital. *American Economic Review*, 94(5):1682–1692, 2004.

Pauli Murto and Juuso Välimäki. Learning and information aggregation in an exit game. *The Review of Economic Studies*, 78(4):1426–1461, 2011.

Mancur Olson. The logic of collective action: Public goods and the theory of groups. 1971.

Julien Prat. Dynamic contracts and learning by doing. *Mathematics and Financial Economics*, 9(3):169–193, 2015.

Julien Prat and Boyan Jovanovic. Dynamic contracts when the agent's quality is unknown. *Theoretical Economics*, 9(3):865–914, 2014.

Canice Prendergast. The provision of incentives in firms. *Journal of economic literature*, 37(1):7–63, 1999.

Roy Radner. Repeated principal-agent games with discounting. *Econometrica*, 53(5): 1173–1198, 1985.

Roy Radner, Roger Myerson, and Eric Maskin. An example of a repeated partnership game with discounting and with uniformly inefficient equilibria. *The Review of Economic Studies*, 53(1):59–69, 1986.

David Rahman. The power of communication. *American Economic Review*, 104(11): 3737–51, 2014.

Dinah Rosenberg, Eilon Solan, and Nicolas Vieille. Social learning in one-arm bandit problems. *Econometrica*, 75(6):1591–1611, 2007.

Tomasz Sadzik and Ennio Stacchetti. Agency models with frequent actions. *Econometrica*, 83(1):193–237, 2015.

Yuliy Sannikov. Games with imperfectly observable actions in continuous time. *Econometrica*, 75(5):1285–1329, 2007.

Yuliy Sannikov. Moral hazard and long-run incentives. *Unpublished working paper, Princeton University*, 2014.

Yuliy Sannikov and Andrzej Skrzypacz. Impossibility of collusion under imperfect monitoring with flexible production. *American Economic Review*, 97(5):1794–1823, 2007.

Yuliy Sannikov and Andrzej Skrzypacz. The role of information in repeated games with frequent actions. *Econometrica*, 78(3):847–882, 2010.

Jean Tirole. *The theory of industrial organization.* MIT press, 1988.

Andrew Von Nordenflycht. What is a professional service firm? toward a theory and taxonomy of knowledge-intensive firms. *Academy of management Review*, 35(1):155–174, 2010.

Iván Werning. Moral hazard with unobserved endowments: A recursive approach. *Unpublished working paper, University of Chicago*, 2001.

Noah Williams. Persistent private information. *Econometrica*, 79(4):1233–1275, 2011.

Eyal Winter. Incentives and discrimination. *American Economic Review*, 94(3):764–773, 2004.

Jiongmin Yong and Xun Yu Zhou. *Stochastic controls: Hamiltonian systems and HJB equations*, volume 43. Springer Science & Business Media, 1999.

A Appendix: Proofs

A.1 Proofs of Propositions 2, 6, and 7.

Proof of Proposition 2. The proof can be split in two parts. First, we establish that for an arbitrary pair of symmetric strategies $\{a_t, a_t\}$, relational capital $\{w_t\}$ follows a process (9), for some L^2 process $\{I_t\}$ and a martingale $\{M_t^w\}$ orthogonal to $\{Y_t\}$. The proof follows similar steps as Proposition 1 in Sannikov [2007]. We derive the representation for the relational capital process in (9) in the second step.

The process $\left\{Y_t - \int_0^t \overline{\mu}_s ds\right\}$, scaled by σ_Y, is a Brownian Motion, and the process $\widetilde{w}_t = \int_0^t e^{-rs}\left(a_s - c(a_s)\right) ds + e^{-rt} w_t$ is a martingale. Since efforts, and so $\widetilde{w}_t$ are bounded, it follows from Proposition 3.4.14 in Karatzas [1991] (of which the Martingale Representation Theorem is a special case, when the filtration $\mathcal{F}_t$ is generated only by the process of profits) that $\widetilde{w}_t$ equals $\int_0^t e^{-rs} I_s \left(dY_s - \overline{\mu}_s ds\right) + M_t^w$, for an appropriate $\{I_t\}$ and a martingale $\{M_t^w\}$. Differentiating and equating both expressions for $\widetilde{w}_t$ yields the representation.

Conversely, for a bounded process $\{v_t\}$ that satisfies (9), define the process $\widetilde{v}_t = \int_0^t e^{-rs}\left(a_s - c(a_s)\right) ds + e^{-rt} v_t$, together with $\widetilde{w}_t$ as above. Both $\{\widetilde{v}_t\}$ and $\{\widetilde{w}_t\}$ are bounded martingales and so, as their values agree at infinity, they agree after every history. It follows that the processes $\{v_t\}$ and $\{w_t\}$ are the same. This establishes the first step.

Let us now evaluate the marginal benefit of effort, and the marginal relational benefit of effort F_τ in particular. Consider the Brownian Motion $\sigma_Y^{-1}\left\{Y_t - \int_0^t \overline{\mu}_s ds\right\}$. It follows from Girsanov's Theorem that the change in the underlying density measure of the output paths induced by the change in expected fundamentals from $\overline{\mu}_\tau$ to $\overline{\mu}_\tau^{dev} = \overline{\mu}_\tau + \varepsilon(r+\alpha)$ is

$$\Gamma_t^\varepsilon = e^{-\frac{1}{2}\int_\tau^t \frac{\left(\overline{\mu}_s^{dev} - \overline{\mu}_s\right)^2}{\sigma_Y^2} ds + \int_\tau^t \frac{\overline{\mu}_s^{dev} - \overline{\mu}_s}{\sigma_Y} \frac{dY_s - \overline{\mu}_s ds}{\sigma_Y}},$$

for $t > \tau$, where $\{\overline{\mu}_s\}_{s \geq \tau}$ and $\{\overline{\mu}_s^{dev}\}_{s \geq \tau}$ are the associated paths of estimates, defined in (2), with $\overline{\mu}_s^{dev} - \overline{\mu}_s = \varepsilon e^{-(\alpha+\gamma)(s-\tau)}$, $s > \tau$. The relational capital at time τ thus changes to

$$\mathbb{E}_{\tau}^{\{a_t,a_t\}}\left[\int_{\tau}^{\infty} e^{-r(t-\tau)}\Gamma_t^{\varepsilon}\left(a_t - c(a_t)\right)dt\right].$$

Since

$$\frac{\partial}{\partial\varepsilon}\Gamma_t^{\varepsilon}\bigg|_{\varepsilon=0} = (r+\alpha)\int_{\tau}^{t} e^{-(\alpha+\gamma)(s-\tau)}\frac{dY_s - \overline{\mu}_s ds}{\sigma_Y},$$

it follows that

$$\begin{aligned}
F_\tau &= \frac{\partial}{\partial\varepsilon}\mathbb{E}_{\tau}^{\{a_t,a_t\}}\left[\int_{\tau}^{\infty} e^{-r(t-\tau)}\Gamma_t^{\varepsilon}\left(a_t - c(a_t)\right)dt\right] \\
&= (r+\alpha)\,\mathbb{E}_{\tau}^{\{a_t,a_t\}}\left[\int_{\tau}^{\infty} e^{-r(t-\tau)}\left(a_t - c(a_t)\right)\left(\int_{\tau}^{t} e^{-(\alpha+\gamma)(s-\tau)}\frac{dY_s - \overline{\mu}_s ds}{\sigma_Y}\right)dt\right] \\
&= (r+\alpha)\,\mathbb{E}_{\tau}^{\{a_t,a_t\}}\left[\int_{\tau}^{\infty}\left(\int_{t}^{\infty} e^{-r(s-t)}\left(a_s - c(a_s)\right)ds\right)e^{-(r+\alpha+\gamma)(t-\tau)}\frac{dY_t - \overline{\mu}_t dt}{\sigma_Y}\right],
\end{aligned}$$

where the last equality follows from the change of integration.

Intuitively, in the last integral above, the inside integral corresponds to the forward looking relational capital, which is then multiplied by a Brownian innovation, scaled by the discounted impact of shifted (expected) fundamentals. The correlation between the relational capital and the Brownian innovation equals I_t, from the representation of the relational capital. This yields F_τ as the expected discounted integral of I_t.

Formally, for $\tau' \geq \tau$,

$$\begin{aligned}
\mathbb{E}_{\tau'}^{\{a_t,a_t\}}\left[F_\tau\right] &= (r+\alpha)\,\mathbb{E}_{\tau'}^{\{a_t,a_t\}}\left[\int_{\tau}^{\infty}\left(\int_{t}^{\infty} e^{-r(s-t)}\left(a_s - c(a_s)\right)ds\right)e^{-(r+\alpha+\gamma)(t-\tau)}\frac{dY_t - \overline{\mu}_t dt}{\sigma_Y}\right] \\
&= (r+\alpha)\left[\int_{\tau}^{\tau'}\left(\int_{t}^{\tau'} e^{-r(s-t)}\left(a_s - c(a_s)\right)ds\right)e^{-(r+\alpha+\gamma)(t-\tau)}\frac{dY_t - \overline{\mu}_t dt}{\sigma_Y}\right] \\
&\quad + (r+\alpha)\,w_{\tau'} \times \left[\int_{\tau}^{\tau'} e^{-r(\tau'-t)}e^{-(r+\alpha+\gamma)(t-\tau)}\frac{dY_t - \overline{\mu}_t dt}{\sigma_Y}\right] + e^{-(r+\alpha+\gamma)(\tau'-\tau)}F_{\tau'}
\end{aligned}$$

is a martingale, as a function of τ'. Using the representation of the relational capital established above, the drift of this martingale equals

$$\begin{aligned}
&(r+\alpha)\left[e^{-(r+\alpha+\gamma)(\tau'-\tau)}I_{\tau'} + \left(\left(a_{\tau'} - c(a_{\tau'})\right) - rw_{\tau'}\right)\int_{\tau}^{\tau'} e^{-r(\tau'-t)}e^{-(r+\alpha+\gamma)(t-\tau)}\frac{dY_t - \overline{\mu}_t dt}{\sigma_Y}\right] \\
&\qquad + \frac{d}{dt}e^{-(r+\alpha+\gamma)(\tau'-\tau)}F_{\tau'},
\end{aligned}$$

where the first term is the covariance of the Brownian increments of $(r+\alpha)w_{\tau'}$ and of the bracketed stochastic intergral in the last line. Integrating over $[\tau,\infty)$ and taking expectation at time τ yields

$$0=(r+a)\,\mathbb{E}_{\tau}^{\{a_t,a_t\}}\left[\int_{\tau}^{\infty}e^{-(r+\alpha+\gamma)(t-\tau)}I_t\right]-F_{\tau}.$$

Using Proposition 3.4.14 from Karatzas [1991] one more time, F_τ satisfies the above equation precisely when it can be represented as in (9).

Finally, since effort increases fundamentals by $(r+\alpha)dt$, and given the decomposition of the continuation value as in (6), the effort process is a local SSE exactly when a_t satisfies $a_t=a(F_t)$ (see e.g. the Verification Theorem in Yong and Zhou [1999] Ch.3.2). This establishes the proof.

Proof of Proposition 6. Note that the boundary condition (16) can be satisfied in two ways. The first line of (16) is equivalent to $I\left(w^{\partial}\right)\left(r+\alpha+\frac{F''(w)}{2}\sigma_Y^2 I(w^{\partial})\right)=0$, which can hold either when $I\left(w^{\partial}\right)=0$, or $I(w^{\partial})=-2\frac{r+\alpha}{\sigma_Y^2F''(w)}>0$. Construction of a local SSE that achieves the boundary in the case $I(w^{\partial})>0$, when relational capital "escapes" the interval $[\underline{w},\overline{w}]$, requires an additional step, as we detail below.

First, we extend the functions F and I beyond the boundary points w^{∂}, at which condition (16) is satisfied with $I\left(w^{\partial}\right)>0$. Consider a boundary point $w^{\partial}=\overline{w}$ and $rw^{\partial}-\left(a(F(w^{\partial}))-c(a(F(w^{\partial})))\right)<0$. We use the Implicit Function Theorem to extend function F to a point $\overline{\overline{w}}>\overline{w}$, so that conditions (16) and $F''(w)<0$ hold on $\left[\overline{w},\overline{\overline{w}}\right]$. We also extend I continuously to the interval $\left[\overline{w},\overline{\overline{w}}\right]$ with $I(w)=-2\frac{r+\alpha}{\sigma_Y^2F''(w)}>0$, so that F and I satisfy the equation (15) on $\left[\overline{w},\overline{\overline{w}}\right]$. In words, on the interval $\left[\overline{w},\overline{\overline{w}}\right]$ the relational incentives can be provided in two ways: they can either consist entirely of the discounted future relational incentives, with zero flow, or by providing inefficiently high flow of relational incentives I. The extension to the interval $\left[\underline{\underline{w}},\underline{w}\right]$ in the case of $w^{\partial}=\underline{w}$ is analogous.

Fix $w_0\in[\underline{w},\overline{w}]$. We first construct a process $\{w_t\}$ of continuation values that satisfies the stochastic equation (9). Let τ^{∞} be the stopping time when $\{w_t\}$ reaches a boundary point that is a local SSE. Moreover, define a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}_+}$ such that $\tau_0 = 0$; for n odd, $\tau_n \geq \tau_{n-1}$ is the stopping time when $\{w_t\}$ reaches either of the new, "outside" boundary points $\{\underline{\underline{w}}, \overline{\overline{w}}\}$; and for $n > 0$ even, $\tau_n \geq \tau_{n-1}$ is the stopping time when $\{w_t\}$ reaches either of the original "inside" boundary points $\{\underline{w}, \overline{w}\}$. For times $t \in [\tau_n, \tau_{n+1})$ with n even and $t < \tau^\infty$ we let $\{w_t\}$ be the weak solution to (9), with $I_t = I(w_t)$ and $\{M_t^w\} = 0$, starting at w_{τ_n}. Existence of a weak solution follows from the continuity of it's drift (which is a consequence of continuity of F and action defined via (8)) and volatility I (see e.g. Karatzas [1991], Theorem 5.4.22). For times $t \in [\tau_n, \tau_{n+1})$ with n odd and $t < \tau^\infty$ we let $\{w_t\}$ be the weak solution to (9), with $I_t = 0$ and $\{M_t^w\} = 0$, starting at w_{τ_n}. In words, the process $\{w_t\}$ has positive volatility until it reaches an "outside" boundary point in $\{\underline{\underline{w}}, \overline{\overline{w}}\}$, after which it drifts "inside" till it reaches the "inside" boundary point in $\{\underline{w}, \overline{w}\}$, when it resumes with the positive volatility, and so on.

It follows from Ito's formula that before τ^∞ the process $F_t = F(w_t)$, satisfies the differential equation in (9), with $\{M_t^F\} = 0$ and $J_t = F'(w_t) \times I(w_t)$. Since both w_t and F_t are bounded, the transversality conditions are satisfied. Finally, we may extend the processes $\{w_t\}, \{I_t\}, \{F_t\}$ and $\{J_t\}$, together with martingales $\{M_t^w\}$ and $\{M_t^F\}$ beyond τ^∞ by letting them follow a local SSE that achieves $(w_{\tau^\infty}, F(w_{\tau^\infty}))$. Then the processes satisfy conditions of Proposition 2.

Proof of Proposition 7. Fix (w_0, F_0) with $w_0 \in (\underline{w}, \overline{w})$ and $F^\lambda(w_0) < F_0 < E(w_0)$, together with a local SSE that achieves it, and let $\{w_t\}$ and $\{F_t\}$ be the processes of relational capital and relational incentives it gives rise to. Define $D(w_t, F_t)$ as the distance of F_t from the solution F^λ of the differential equation (17),

$$D(w_t, F_t) = F_t - F^\lambda(w_t).$$

Using Ito's lemma together with the Proposition 2, at any time t when $D(w_t, F_t) \in [0, \delta]$, the drift of the process $D(w_t, F_t)$ equals, for appropriate process $\{I_t\}$,

$$\begin{aligned}\frac{\mathbb{E}\left[dD(w_t, F_t)\right]}{dt} &= (r+\alpha+\gamma)\, F_t - (r+\alpha) I_t - F^{\lambda\prime}(w_t) \times (rw_t - (a(F_t) - c(a(F_t)))) \\ &- \frac{F^{\lambda\prime\prime}(w)\left[\sigma_Y^2 I_t^2 + d\left\langle M_t^w\right\rangle\right]}{2} \\ &\geq (r+\alpha+\gamma)\, F_t - (r+\alpha) I_t - F^{\lambda\prime}(w_t) \times \Big(rw_t - \Big(a(F^\lambda(w_t)) - c(a(F^\lambda(w_t)))\Big)\Big) \\ &- \frac{F^{\lambda\prime\prime}(w)\left[\sigma_Y^2 I_t^2 + d\left\langle M_t^w\right\rangle\right]}{2} - \frac{\lambda}{2} \\ &\geq (r+\alpha+\gamma)\Big(F_t - F^\lambda(w_t)\Big) + \lambda - \frac{\lambda}{2} > (r+\alpha+\gamma) \times D(w_t, F_t),\end{aligned} \tag{18}$$

The first inequality holds because $\left|F^{\lambda\prime}(w_t)\right| \leq 1/\lambda$, functions a and c are Lipschitz continuous and $D\left(w_t, F_t\right) \in [0, \delta]$, where δ is assumed to be sufficiently small. The second inequality follows because F^λ satisfies

$$\begin{aligned}(r+\alpha+\gamma) F^\lambda(w) =& \max_I \Big\{(r+\alpha) I + F^{\lambda\prime}(w)\Big(rw - \Big(a(F^\lambda(w)) - c(a(F^\lambda(w)))\Big)\Big) \\ &+ \frac{F^{\lambda\prime\prime}(w)\sigma_Y^2}{2} I^2\Big\} + \lambda,\end{aligned} \tag{19}$$

F^λ is concave, and $d\left\langle M_t^w\right\rangle$ is positive. Let τ be the stopping time of the process $D(w_t, F_t)$ hitting zero. Due to $D(w_0, F_0) > 0$ and inequality (18), it follows that there is a finite time T such that $E\left[D(w_T, F_T) | \tau \geq T\right] > \delta$. On the other hand, since

$$\begin{aligned}E\left[D(w_{\min\{T,\tau\}}, F_{\min\{T,\tau\}})\right] &= P\left(\tau \geq T\right) \times E\left[D(w_T, F_T) | \tau \geq T\right] \\ &+ P\left(\tau < T\right) \times E\left[D(w_\tau, F_\tau) | \tau < T\right] \\ &= P\left(\tau \geq T\right) \times E\left[D(w_T, F_T) | \tau \geq T\right],\end{aligned}$$

and the expectation is positive, it follows that $P\left(\tau \geq T\right) > 0$. This establishes that $D(w_T, F_T)$ exceeds δ with positive probability, contradiction.

A.2 Proof of Theorem 1

Let $\mathcal{E}$ to be the set of pairs of relational capital and relational incentives, (w, F), achievable in local SSE, and let the partial function $E : \mathbb{R} \to \mathbb{R}$ parametrize the upper boundary of this set, as a function of w.

We begin the proof of the Theorem with the following two technical lemmas. We define the efficient level of relational capital as $w_{EF} = \frac{1}{r}\left(a_{EF} - c(a_{EF})\right)$, for the efficient effort level a_{EF}, with $c'(a_{EF}) = 1$. Let also $\underline{\mathrm{F}}$ be the lower arm of the parabola, which is the locus of the feasible relational capital-incentives pairs (w, F) that can be achieved by symmetric play in a stage game, satisfying $rw = a(\underline{F}) - c(a(\underline{F}))$; see Figure 1.

Lemma 1 *The set $\mathcal{E}$ is convex and $w^* \leq w_{EF}$. Moreover, the upper boundary E satisfies $E(w) \geq \underline{F}(w) > 0,\ w \in (0, w^*)$.*

Proof. Convexity is immediate from the possibility of public randomization, and the inequality $w \leq w_{EF}$ follows from the definitions. Finally, suppose by the way of contradiction that there exists w, $0 \leq w < w^*$, such that $E(w) < \underline{F}(w)$. Note that at w the slope of E is smaller than the slope of $\underline{F}$: otherwise, the repeated static Nash point $(0, 0)$, belonging to the graph of the convex function $\underline{F}$, and the convex set $\mathcal{E}$ would not overlap. This implies that E is bounded away below $\underline{F}$ to the right of w, and so in any local SSE the relational capital has drift bounded away above zero, as long as $w_t \geq w$ (see (9)). The possibility of escape of relational capital beyond w^* establishes the contradiction. ■

Lemma 2 *Let $F, E : [\underline{w}, \overline{w}) \to \mathbb{R}$ be two concave functions such that*

i) $E \leq F$,

ii) $E(\underline{w}) = F(\underline{w})$ *and* $E'_+(\underline{w}) = F'_+(\underline{w})$,

iii) $F''_+(\underline{w})$ *exists*

Then either $E''_+(\underline{w})$ exists and equals $F''_+(\underline{w})$ or there is G with $G(\underline{w}) = E(\underline{w})$, $G'_+(\underline{w}) = E'_+(\underline{w})$ and $G''_+(\underline{w}) < F''_+(\underline{w})$ such that $E \leq G$ in a right neighborhood of $\underline{w}$.

Proof. Suppose that $E''_+(\underline{w})$ does not exist or is not equal to $F''_+(\underline{w})$. From i), this means that there is a $\varepsilon > 0$ and a decreasing sequence $\{w_n\} \to \underline{w}$ such that

$$E(w_n) \leq F(\underline{w}) + F'_+(\underline{w}) \times (w_n - \underline{w}) + \left(F''_+(\underline{w}) - \varepsilon\right) \times (w_n - \underline{w})^2 .$$

However, concavity of E implies that the above inequality holds not only for the sequence $\{w_n\}$ but in a right neighborhood of $\underline{w}$. This implies the result, with $G(w) = F(w) - \varepsilon(w - \underline{w})^2$ in a neighborhood of $\underline{w}$. ■

The proof of Theorem 1 rests on the following four propositions. Relying on Propositions 6 and 7, as well as the above two lemmas, they establish that: (i) the boundary points $(w, E(w))$, for $w > 0$, may not be generated by solely defered incentives from the future and require strictly positive volatility of relational capital, or flow of incentives; (ii) the boundary E is differentiable; (iii) given any boundary point $(w, E(w))$ and a tangent vector E', the solution of HJB equation (11) with those boundary conditions must locally lie weakly above the boundary E as well as (iv) weakly below the boundary E.

The propositions thus establish that in the range where the boundary $E(w)$ is strictly positive, it must satisfy the HJB equation (11). The proof is then concluded by establishing the boundary conditions (12).

Proposition 8 *If $(1, E')$ is a tangent vector at $(w_0, E(w_0))$, with $w_0 > 0$, then*

$$(r + \alpha + \gamma)E(w_0) > E' \times (rw_0 - [a(E(w_0)) - c(a(E(w_0)))]) . \tag{20}$$

Proof. Pick $w_0 > 0$; it follows from Lemma (1) that $E(w_0) > 0$. If the drift term is zero, $rw_0 - (a(E(w_0)) - c(a(E(w_0)))) = 0$, then (20) holds. Suppose then that the drift is strictly negative, $rw_0 - (a(E(w_0)) - c(a(E(w_0)))) < 0$ (when the inequality is reversed the proof is analogous), and such that inequality (20) fails. Assume also that $(w_0, E(w_0))$ is achieved by a local SSE, as opposed to being a limit of local SSE pairs – an assumption that we relax at the end of the proof.

Let $\overline{E}' \geq E'$ be such that (20) holds with equality, with $\overline{E}'$ in place of E'. Consider the function F defined over $[w_0, w']$, where w' is in the right neighborhood of w_0, such that F satisfies (20) with equality, with initial contition $(F(w_0), F'(w_0)) = (E(w_0), \overline{E}')$, and such that $w - (a(F(w)) - c(a(F(w)))) < 0$ for all $w \in [w_0, w']$. F is the solution of the implicit function second order ordinary differential equation.

Since $(w_0, F(w_0))$ is achieved by a local SSE and the boundary condition (12) holds at w', the function F satisfies conditions of Proposition 6, together with $I \equiv 0$. Consequently, there are local SSE that achieve every pair in its graph, and so the function lies below the boundary, $F(w) \leq E(w)$, $w \in [w_0, w']$. (Note that it follows that the inequality $(r+\alpha+\gamma)E(w_0) < E' \times (rw_0 - [a(E(w_0)) - c(a(E(w_0)))])$ is impossible, or else $\overline{E}' > E'$ and F lies above E.)

Consider now a strictly concave quadratic function G^* defined in the right neighborhood of w_0 with $(G^*(w_0), G^{*\prime}(w_0)) = (E(w_0), E'(w_0)$ and $G^*(w) < F(w)$ for $w > w_0$. The function satisfies

$$(r+\alpha+\gamma)\, G^*(w) < G^{*\prime}(w)\,(rw - [a(G^*(w)) - c(a(G^*(w)))]) - \frac{(r+\alpha)^2}{2\sigma_Y^2 G^{*\prime\prime}}, \tag{21}$$

in a right neighborhood of w_0. But then, by increasing slightly $G^{*\prime}(w_0)$, we may construct a quadratic function G over an inteval $[w_0, \overline{w}]$ that also satisfies (21), together with $G(w_0) = E(w_0)$, $G'(w_0) > E'(w_0)$, and $G(\overline{w}) < E(\overline{w})$. There exists then a function $I : [w_0, \overline{w}] \to \mathbb{R}$, with $I(w) > -\frac{(r+\alpha)^2}{\sigma_Y^2 G''}$, such that

$$(r+\alpha+\gamma)\, G(w) = I(w) + G'(w)\,(rw - [a(G(w)) - c(a(G(w)))]) + \frac{G''\sigma_Y^2}{2} I^2. \;\; w \in [w_0, \overline{w}]$$

Applying Proposition 2, each point $(w, G(w))$, for $w \in [w_0, \overline{w}]$, can be achieved by a local SSE. Since $G'(w_0) > E'(w_0)$, this yields the desired contradiction.

Finally, when $(w_0, E(w_0))$ is not achieved by a local SSE, the result follows for the functions F, G^*, and G defined analogously as before, but with $F(w_0) = G^*(w_0) = G(w_0) = E(w_0) - \varepsilon$, for sufficiently small $\varepsilon > 0$. ■

Consider now the HJB equation (11), written as $F''(w) = \mathcal{F}(w, F, F')$. Proposition 8 implies that the right hand side of this equation is well defined and is Lipschitz continuous in the neighborhood of the points $(w_0, E(w_0), E')$, for any w_0 in $(0, w^*)$ and a tangent vector $(1, E')$, with $F'' < 0$. The following corollary is used repeatedly in the proof of the theorem:

Corollary 3 *The solution of the HJB equation (11) exists and depends continuously on* *the initial parameters in the neighborhood of the boundary condition* $(w_0, E(w_0), E')$*, for any* w_0 *in* $(0, w^*)$ *and a tangent vector* $(1, E')$.

Proposition 9 *The upper boundary* E *of the set of relational capital and relational incentives achievable in a local SSE is differentiable in* $(0, w^*)$.

Proof. Suppose to the contrary that $(w_0, E(w_0))$ is a kink. If follows from Proposition 8 that for any tangent vector $(1, E')$ at $(w_0, E(w_0))$

$$(r + \alpha + \gamma)E(w_0) > E' \times (rw_0 - [a(E(w_0)) - c(a(E(w_0)))]) .$$

Continuous dependence on the initial parameters implies that there exists $\lambda > 0$ such that $F^{\lambda *}$ solving (17) with the same initial conditions is strictly above curve E in a neighborhood of w_0 (excluding point w_0). Invoking the continuous dependence once again, this time shifting the initial condition $(w_0, E(w_0), E')$ down to $(w_0, E(w_0) - \delta, E')$, for $0 < \delta << \lambda$, we construct a function F^{λ} that satisfies the conditions of Lemma 7, yielding a contradiction. ■

Proposition 10 *For any* w_0 *in* $(0, w^*)$*, the solution* F *to the differential equation (11) with initial condition* $(w_0, E(w_0), E'(w_0))$ *is weakly above the curve* E *in a neighborhood of* w_0.

Proof. Suppose to the contrary that $F < E$ in, say, the right neighborhood of w_0 (the case of the left neighborhood is analogous). From continuous dependance on the initial parameters, there are $\varepsilon, \delta > 0$ such that that the solution $\widetilde{F}$ of (11) with initial conditions $(w_0, E(w_0) - \delta, E'(w_0) + \varepsilon)$ crosses above and then comes back to E, meaning $\widetilde{F}(w_1) > E(w_1)$ and $\widetilde{F}(w_2) < E(w_2)$ for some $w_2 > w_1 > w_0$. But then the function $\widetilde{F}$ defined on $[w_0, w_2]$ satisfies conditions of Proposition 6, and so its graph is achievable by local SSE. This yields a contradiction. ■

Proposition 11 *For any w_0 in $(0, w^*)$, the solution F to the differential equation (11) with initial condition $(w_0, E(w_0), E'(w_0))$ is weakly below the curve E in a neighborhood of w_0.*

Proof. Let F satisfy (11) with initial conditions $(w_0, E(w_0), E'(w_0))$ and suppose that either $E''_+(w_0)$ does not exist, or $E''_+(w_0) \neq F''_+(w_0)$ (the case of left second derivative is analogous). Propositions 9 and 10 establish that the conditions of Lemma 2 are satisfied at w_0, and so in the right neighborhood of w_0 E is bounded above by $F(w) - \bar{\varepsilon}(w - w_0)^2$, for appropriate $\bar{\varepsilon} > 0$. Continuous dependence on initial parameters implies that there exists $\varepsilon > 0$ such that $F^{\lambda *}$ solving (17) with the same initial conditions $(w_0, E(w_0), E'(w_0))$ as F has second derivative at w_0 strictly larger than $F''(w_0) - \bar{\varepsilon}$ and is strictly above curve E in a right neighborhood of w_0 (excluding point w_0). Invoking the continuous dependence once again, this time turning the initial condition $(w_0, E(w_0), E'(w_0))$ right to $(w_0, E(w_0), E'(w_0) - \delta)$, for $0 < \delta << \lambda$, we construct a function F^{λ} that satisfies the conditions of Lemma 7, yielding a contradiction. ■

The proof so far established that the boundary E satisfies the HJB equation (11) on $(0, w^*)$. To conclude the proof of the theorem, it remains to establish the boundary conditions (12).

1. $E(0) = 0$. Strictly positive relational incentives at zero in a local SSE would imply that the expected discounted efforts by each agent are strictly positive; consequently, a deviation to zero effort always would yield a nonzero relational capital to a partner, contradiction.

2. $\lim_{w \uparrow w^*} E(w) = \underline{F}(w^*)$. i) Lemma 1 shows that $\lim_{w \uparrow w^*} E(w) < \underline{F}(w^*)$ is impossible. ii) If $\lim_{w \uparrow w^*} E(w) \in \big(\underline{F}(w_0), \overline{F}(w_0)\big)$, then, using Proposition 2, it would be possible to extend the solution to the right, with $I(w) = 0$ for $w > w^*$, contradiction. iii) If $\lim_{w \uparrow w^*} E(w) = \overline{F}(w_0)$ then, whether E approaches $\overline{F}$ from above or below, the differential equation (11) would be violated in the left neighborhood of w_*. iv) If $\lim_{w \uparrow w^*} E(w) > \overline{F}(w_0)$, then relational capital in any local SSE achieving points close to $(w^*, \lim_{w \uparrow w^*} E(w))$ has strictly positive drift, bounded away from zero. This would lead to the escape of w to the right of w^*, with positive probability.

3. $\lim_{w \uparrow w^*} E''(w) = -\infty$. When the condition is violated, then $I^*(w)$ is continuous and strictly positive close to w^*. The proof of the theorem so far establishes that E is C^2 and satisfies the differential equation (11). Given this regularity, standard verification theorem techniques establish that the equilibria achieving $(w, E(w))$, $w < w^*$, must use the optimal flow of relational incentives $I^*(w)$ a.e. (see Yong and Zhou [1999]); when $(w, E(w))$ is unattainable, the same is true for (w, F) in the limit, with F approaching $E(w)$. This, however, leads to the relational capital escaping to the right of w^*, with positive probability.

A.3 Proof of Proposition 3

Part i) The proof strategy is to construct a a C^2 function $F : [0, \overline{w}] \to R$ that satisfies the differential inequality

$$(r+\alpha+\gamma)F(w) \leq F'(w) \times (rw - [a(F(w)) - c(a(F(w)))]) - \frac{(r+\alpha)^2}{2\sigma_Y^2 F''(w)}, \tag{22}$$

together with the left boundary condition $F(0) = 0$ (achieveable by the Markov equilibrium), and the right boundary condition (16). Given such an F, it is always possible to find an $I(w) \geq -\frac{r+\alpha}{\sigma_Y^2 F''(w)}$ for which the equation (15) in Proposition 6 holds at every $w \in [0, \overline{w}]$.

Given the quadratic cost of effort $c(a) = \frac{a}{2} + \frac{C}{2}a^2$, the flow payoffs (given interior efforts) satisfy

$$a(F) - c(a(F)) = \frac{F(w)}{2C}(1 - F(w)),$$

and also $\underline{F}'(0) = 2Cr$ (see Section 3.2). We will construct a curve F over $[0, \overline{w}]$, with $\overline{w} = \delta/r = \frac{1}{16Cr}$, constant second derivative and with the right boundary condition

$$F(\overline{w}) = \frac{1}{2} > \underline{F}(\overline{w}),$$

as well as

$$F'(\overline{w}) = \frac{(r+\alpha+\gamma)F(\overline{w})}{r\overline{w} - \frac{F(\overline{w})}{2C}(1 - F(\overline{w}))} = \frac{\frac{1}{2}(r+\alpha+\gamma)}{\delta - \frac{1}{8C}} = -4C(r+\alpha+\gamma),$$

so that the first equation in (16) is satisfied at $\overline{w}$; the second equation follows from $F(\overline{w}) \in \left(\underline{F}(\overline{w}), \overline{F}(\overline{w})\right)$.

The constant second derivative D is pinned down by

$$\begin{aligned}
F\left(\overline{w}\right) &= \int_0^{\overline{w}} F'(x)dx = \int_0^{\overline{w}} \left[F'(\overline{w}) - D\left(\overline{w} - x\right)\right]dx \\
&= F'(\overline{w}) \times \frac{\delta}{r} - \frac{D}{2}\left(\frac{\delta}{r}\right)^2, \\
\frac{1}{D} &= \frac{1}{2}\frac{1}{F'(\overline{w}) \times \frac{\delta}{r} - F\left(\overline{w}\right)}\left(\frac{\delta}{r}\right)^2 = \frac{1}{2}\frac{1}{-\frac{1}{4}(r+\alpha+\gamma) \times \frac{1}{r} - \frac{1}{2}}\left(\frac{\delta}{r}\right)^2 \\
&= -\frac{2}{2+r+\alpha+\gamma}\left(\frac{\delta}{r}\right)^2.
\end{aligned}$$

It follows that, for all $w \in [0, \overline{w}]$,

$$\begin{aligned}
F(w) &\leq \frac{1}{2} + 4C(r+\alpha+\gamma) \times \frac{\delta}{r} \leq \frac{r+\alpha+\gamma}{r}, \qquad (23)\\
|F'(w)| \leq F'(0) &\leq |F'\left(\overline{w}\right)| + \frac{F\left(\overline{w}\right) - 0}{|F'\left(\overline{w}\right)|}|D| = 4C(r+\alpha+\gamma) + \frac{2+r+\alpha+\gamma}{r+\alpha+\gamma}16Cr^2, \\
rw - \frac{F(w)}{2C}(1-F(w)) &\geq -\frac{1}{8C}, \\
(r+\alpha+\gamma)F(w) - F'(w)\left(rw - \frac{F(w)}{2C}\left(1-F(w)\right)\right) &\leq \frac{(r+\alpha+\gamma)^2}{r} \\
&\quad + \frac{r+\alpha+\gamma}{2} + \frac{2+r+\alpha+\gamma}{r+\alpha+\gamma}2r^2, \\
-\frac{(r+\alpha)^2}{2\sigma_Y^2 D} = \frac{(r+\alpha)^2}{2\sigma_Y^2}\frac{2}{2+r+\alpha+\gamma}\left(\frac{1}{16Cr}\right)^2 &\geq \frac{2}{512\sigma_Y^2 C^2(2+r+\alpha+\gamma)},
\end{aligned}$$

where we also assume that the bound A is high enough so that the efforts are interior,

$$A \geq C\frac{r+\alpha+\gamma}{r} \geq C\max_w F\left(w\right) \geq \max_w a\left(F\left(w\right)\right).$$

The last two inequalities in (23) establish that inequality (22) is satisfied, and so nontrivial local SSE exist, as long as

$$\frac{(r+\alpha+\gamma)^2}{r} + \frac{r+\alpha+\gamma}{2} + 2r^2\frac{2+r+\alpha+\gamma}{r+\alpha+\gamma} \leq \frac{2}{512\sigma_Y^2 C^2(2+r+\alpha+\gamma)},$$

or

$$(r+\alpha+\gamma)\frac{2+r+\alpha+\gamma}{2}\left(\frac{r+\alpha+\gamma}{r} + \frac{1}{2} + 2\left(2+r+\alpha+\gamma\right)\left(\frac{r}{r+\alpha+\gamma}\right)^2\right) \leq \frac{1}{512\sigma_Y^2 C^2}. \qquad (24)$$

This establishes the proof of part i) of the proposition.

Results do not depend of normalizing the marginal benefit of effort: The proposition remains true when the effect of action is scaled up by $X > 1$, so that $d\mu_t = X(r+\alpha)(a_t^1 + a_t^2)dt - \alpha\mu_t dt + \sigma_\mu dB_t^\mu$ (for example, when the effect is independent of $r+\alpha$, we have $X = (r+\alpha)^{-1}$). We briefly comment here how the proof of the proposition must be adjusted.

For a fixed $X > 1$ the Markov equilibrium action becomes $a_M^X = \frac{X-1}{2C}$, and, given relational incentives F^X, the locally optimal action $a^X(F^X)$ equals $a_M^X + \frac{F^X}{C}$. The flow of relational capital (flow of equilibrium utility net of Markov equilibrium level) is $Xa^X(F^X) - c(a^X(F^X))$, which equals $\frac{F^X}{2C}\left(X - F^X\right)$; consequently, the HJB equation generalizes from (11) in Theorem 1 to

$$
\begin{aligned}
(r+\alpha+\gamma)F^X(w) &= \max_I \left\{ X(r+\alpha)I + F^{X\prime}(w)\frac{F^X(w)}{2C}\left(X - F(w)\right) + \frac{F^{X\prime\prime}(w)\sigma_Y^2}{2}I^2 \right\} \quad (25)\\
&= F^{X\prime}(w)\frac{F^X(w)}{2C}\left(X - F^X(w)\right) - \frac{X^2(r+\alpha)^2}{2\sigma_Y^2 F^{X\prime\prime}(w)}.
\end{aligned}
$$

For the new parametrization, the construction remains analogous as in the proposition, with $a_{EF}^X = Xw_{EF}^1$, $w_{EF}^X = Xw_{EF}^1$, $\overline{w}^X = X\overline{w}^1$, $F^X(\overline{w}^X) = XF^1(\overline{w}^1)$, $F^{X\prime}(\overline{w}^X) = F^{1\prime}(\overline{w}^1)$, and $F^{X\prime\prime}(w) = \frac{1}{X}F^{1\prime\prime}(w)$. The bounds (23) in the proof change to: $\overline{F^X(w)} \leq X \times \overline{F^1(1)}$, $\overline{|F^{X\prime}(w)|} = \overline{|F^{1\prime}(w)|}$, and $\underline{(rw - (2C^{-1})F^X(w)(X - F^X(w)))} \geq X \times \underline{(rw - (2C^{-1})F^X(w)(X - F^X(w)))}$. Consequently, all the terms in the inequality (22) are bounded by the terms scaled up by X, and the inequality continues to hold.

Part ii) Fix $\underline{w} > 0$. In the proof we show that if the constant in the statement of the proposition is sufficiently high, then $w^* \leq \underline{w}$.

Suppose that $w^* > \underline{w}$. Observe that for all w such that $F'(w) \leq 0$ we have

$$
F(w) \geq \lim_{s \to w^*} F(s) = \underline{F}(w^*) > \underline{F}(\underline{w}) \geq 2Cr\underline{w} =: A, \quad (26)
$$

where the last inequality follows from $\underline{F}(0) = 0$, $\underline{F}'(0) = 2Cr$, and $\underline{F}$ convex. Secondly, recall from Theorem 1 that as w approaches w^* from the left, then $F'(w)$ gets arbitrarily high, and $F''(w)$ arbitrarily low. Finally, note that for any $w > 0$ the drift of the relational capital is uniformly bounded from below by

$$rw-(a(F(w)) - c(a(F(w)))) > -[a(F(w))-c(a(F(w)))] \geq -[a_{EF}-c_{EF}] = -\frac{1}{8C} =: -B. \tag{27}$$

In the first part of the proof we establish that if the constant in the statement of the theorem is sufficiently high, then the the value $F(w^{\#})$ of relational incentives at the point $w^{\#}$ such that $F'(w^{\#}) = 0$ would be arbitrarily high as well. We lead it to contradiction in the second part of the proof.

Fix $\overline{w}$ close to w^*, such that $-F''(\overline{w})$ equals $\varepsilon^{-1} > 0$ sufficiently large, to be determined later. Consider the differential equation

$$(r + \alpha + \gamma)A = -G'(w)B - \frac{(r + \alpha)^2}{2\sigma_Y^2 G''(w)}, \tag{28}$$

together with a boundary condition $G(\overline{w}) = F(\overline{w})$, $G''(\overline{w}) = F''(\overline{w})$, and solved for $w \leq \overline{w}$.

Let $w^{\#\#} < \overline{w}$ be such that $G'\left(w^{\#\#}\right) = 0$. We argue that

$$G'(w) > F'(w), \text{ for all } w \in [w^{\#\#}, \overline{w}]. \tag{29}$$

Indeed, note that F satisfies equation (11), related to (28), but with $F(w)$ in place of A, and $rw - (a(F(w)) - c(a(F(w))))$, in place of $-B$. It follows from (26) and (27) that (29) holds at $w = \overline{w}$. Similarly, suppose $w^{\&}$, $w^{\#} < w^{\&} < \overline{w}$, was the maximal point such that $G'(w^{\&}) \leq F'\left(w^{\&}\right)$. It follows that $F(w^{\&}) > G(w^{\&})$ and $rw^{\&} - \left(a(F(w^{\&})) - c(a(F(w^{\&})))\right) > -B$, and so $G''(w^{\&}) < F''(w^{\&})$. This last inequality contradicts maximality of $w^{\&}$.

Crucially, inequality (29) implies that

$$F(w^{\#}) > G(w^{\#\#}), \tag{30}$$

for the maximal values of the respective functions, with $F'(w^{\#}) = 0$ and $G'(w^{\#\#}) = 0$.

We now compute $G(w^{\#\#})$. The solution to the differential equation (29) takes the form

$$G'(w) = \frac{\sqrt{\varepsilon^2 + 2c(\overline{w} - w)} - d}{c}, \quad G''(w) = -\frac{1}{\sqrt{\varepsilon^2 + 2c(\overline{w} - w)}},$$

for

$$d = 2\left(\frac{\sigma_Y}{r+\alpha}\right)^2 (r+\alpha+\gamma)\,A, \quad c = 2\left(\frac{\sigma_Y}{r+\alpha}\right)^2 B.$$

It follows that

$$\begin{aligned}
w^{\#\#} &= \overline{w} - \frac{d^2-\varepsilon^2}{2c}, \\
G(w^{\#\#}) &= G(\overline{w}) - \int_{w^{\#\#}}^{\overline{w}} G'(w)dw = G(\overline{w}) - \int_{w^{\#\#}}^{\overline{w}} \frac{\sqrt{\varepsilon^2+2c(\overline{w}-w)}-d}{c}dw \\
&= G(\overline{w}) + \frac{d^2-\varepsilon^2}{2c}\frac{d}{c} + \frac{1}{c}\frac{2}{3}\frac{1}{2c}\left[\varepsilon^2+2c(\overline{w}-w)\right]^{3/2}\Big|_{w^{\#\#}}^{\overline{w}} \\
&= G(\overline{w}) + \frac{d^2-\varepsilon^2}{2c}\frac{d}{c} + \frac{\varepsilon^3}{3c^2}\frac{d^2-\varepsilon^2}{2c} - \frac{1}{3c^2}\left(d^2-\varepsilon^2\right)^{3/2} \geq \frac{d^3}{2c^2} - \frac{d^3}{3c^2} = \frac{1}{6}\frac{d^3}{c^2},
\end{aligned}$$

where the last inequality holds when ε is chosen small enough.

Substituting for d, c, and B in the above bound for $G(w^{\#\#})$, and using (30) we have

$$F(w^{\#}) \geq \frac{64}{3}\left(\frac{\sigma_Y}{r+\alpha}\right)^2 (r+\alpha+\gamma)^3 CA^3 =: D. \tag{31}$$

We now derive a contradiction from (31), when D is large enough. Let $w^\circ \in (w^{\#}, \overline{w})$ be such that $F(w^\circ) = \frac{1}{2}D$. Note that when, as we shall suppose,

$$\frac{1}{2}D > 1 = \overline{F}(0) \geq \overline{F}(w), \text{ for all } w \in [0, w_{EF}],$$

then for all $w \in [w^{\#}, w^\circ]$ the drift of relational capital $rw - (a(F(w)) - c(a(F(w))))$ is positive, and so

$$(r+\alpha+\gamma)F(w) < -\frac{(r+\alpha)^2}{2\sigma_Y^2 F''(w)}, \quad w \in [w^{\#}, w^\circ]$$

or

$$-F''(w) < \frac{(r+\alpha)^2}{2\sigma_Y^2(r+\alpha+\gamma)F(w)} \leq \frac{(r+\alpha)^2}{\sigma_Y^2(r+\alpha+\gamma)D}, \quad w \in [w^{\#}, w^\circ]. \tag{32}$$

Summarizing, when $D > 2$ we have

$$\begin{aligned}
\frac{1}{2}D = F(w^{\#}) - F(w^\circ) &< \frac{(r+\alpha)^2}{\sigma_Y^2(r+\alpha+\gamma)D}(w^\circ - w^{\#})^2 \\
&< \frac{(r+\alpha)^2}{\sigma_Y^2(r+\alpha+\gamma)D}\left(\frac{1}{8Cr}\right)^2,
\end{aligned}$$

where the first equality follows from the definition of w°, the first inequality follows from

$F'(w^{\#}) = 0$ and the bound (32), and the lest bound follows from $w^{\circ} - w^{\#} < w_{EF} - 0 = \frac{1}{8Cr}$. Rearranging the last inequality, and substituting for D we have the necessary condition

$$1 > 32D^2C^2r^2\frac{\sigma_Y^2(r+\alpha+\gamma)}{(r+\alpha)^2} = \frac{64^3 2^6}{18}C^{10}r^8\left(\frac{\sigma_Y}{r+\alpha}\right)^6(r+\alpha+\gamma)^7\underline{w}^6, \tag{33}$$

which establishes contradiction, when $r+\alpha+\gamma$ is sufficiently large. This concludes the proof of the proposition.

Results do not depend of normalizing the marginal benefit of effort: As in the case of part i), part ii) of the proposition remains true when the effect of action is scaled up by $X < 1$, so that $d\mu_t = X\,(r+\alpha)\,(a_t^1 + a_t^2)dt - \alpha\mu_t dt + \sigma_\mu dB_t^\mu$ (for example, when the effect is independent of $r+\alpha$, we have $X = (r+\alpha)^{-1}$). We briefly comment here how the proof of the proposition must be adjusted.

Fix $X < 1$; the bounds in the proposition change to $A^X = \frac{1}{X}A^1$, $B^X = X \times B^1$, and the last term in the equation (28) is scaled up by X^2 (see (25)). Consequently, $d^X = \frac{1}{X^3}d^1$, $c^X = \frac{1}{X} \times c^1$, $G^X(w^{X\#\#}) = \frac{1}{X^7}G^1(w^{1\#\#})$. This results in in bounds $-\overline{F^{X\prime\prime}(w)} \leq -\overline{F^{1\prime\prime}(w)} \times X^9$ and, rearranging terms, the right-hand side in the necessary inequality (33) is multiplied by $X^{18} < 1$.

A.4 Proof of Theorem 2

In the following proofs we will need the following result.

Lemma 3 *For any $\varepsilon > 0$ and the function F_ε from Theorem 2*

$$F_\varepsilon \leq \frac{(r+\alpha)^2}{256\sigma_Y^2\,(r+\alpha+\gamma)\,r^2C^2} + 1. \tag{34}$$

Proof. Let $w_0 \in [0, \overline{w}_\varepsilon]$ be the point at which F_ε is maximized, $F_\varepsilon'(w_0) = 0$. For $w \geq w_0$ such that $F_\varepsilon(w) \geq \overline{F}(0) = 1 \geq \overline{F}(w)$, so that the drift of the relational capital

$rw-(a(F_\varepsilon(w))-c(a(F_\varepsilon(w))))$ is positive, we have

$$(r+\alpha+\gamma)\,F_\varepsilon(w)=F_\varepsilon'(w)\,(rw-[a(F_\varepsilon(w))-c(a(F_\varepsilon(w)))])-\frac{(r+\alpha)^2}{2\sigma_Y^2F_\varepsilon''(w)}\leq-\frac{(r+\alpha)^2}{2\sigma_Y^2F_\varepsilon''(w)}, \tag{35}$$

$$-F_\varepsilon''(w)\leq\frac{(r+\alpha)^2}{2\sigma_Y^2\,(r+\alpha+\gamma)},$$

where the equality follows from the fact that $F_\varepsilon''(w)\geq-\frac{r+\alpha}{\sigma_Y^2\varepsilon}$ (otherwse the right hand side would fall short of 1, and so the left hand side). Since $\overline{w}_\varepsilon\leq w_{EF}=1/8rC$, it therefore follows that

$$\begin{aligned}F_\varepsilon(w_0)&\leq F_\varepsilon(w_0)-F_\varepsilon(\overline{w}_\varepsilon)+1\leq\frac{1}{2}\frac{(r+\alpha)^2}{2\sigma_Y^2\,(r+\alpha+\gamma)}\left(\frac{1}{8rC}\right)^2+1\\&=\frac{(r+\alpha)^2}{256\sigma_Y^2\,(r+\alpha+\gamma)\,r^2C^2}+1.\end{aligned}$$

■

The proof of the first part of the Theorem is analogous to the proof of Theorem 1. The optimal policy function implied by 14 is given by

$$\begin{aligned}&I_\varepsilon^*(w)=-\frac{r+\alpha}{\sigma_Y^2F''(w)},\quad\text{if } F_\varepsilon''(w)\geq-\frac{r+\alpha}{\sigma_Y^2\varepsilon}\\&I_\varepsilon^*(w)=\varepsilon,\quad\text{if }-2\frac{r+\alpha}{\sigma_Y^2\varepsilon}<F_\varepsilon''(w)<-\frac{r+\alpha}{\sigma_Y^2\varepsilon}\\&I_\varepsilon^*(w)=0.\quad\text{if } F_\varepsilon''(w)\leq-2\frac{r+\alpha}{\sigma_Y^2\varepsilon}\end{aligned} \tag{36}$$

It is easy to establish that $F_\varepsilon''(0)=-2\frac{r+\alpha}{\sigma_Y^2\varepsilon}$, since with any other value, the equation (14) would be violated around zero. In what follows we establish that if there is w such that $F_\varepsilon''(w)<-2\frac{r+\alpha}{\sigma_Y^2\varepsilon}$, then $F_\varepsilon'(w)<<0$ and $rw-(a(F_\varepsilon(w))-c(a(F_\varepsilon(w))))\approx 0$. We claim that this is enough to establish the proof of the Theorem. Indeed, we may define $\overline{w}_\varepsilon'$ as the first point such that $F_\varepsilon''(\overline{w}_\varepsilon')=-2\frac{r+\alpha}{\sigma_Y^2\varepsilon}$ and $F_\varepsilon''(w)<-2\frac{r+\alpha}{\sigma_Y^2\varepsilon}$ in the right neighborhood of $(\overline{w}_\varepsilon',\overline{w}_\varepsilon]$, for some $\overline{w}_\varepsilon>\overline{w}_\varepsilon'$. Note that, crucially, the policy I_ε^* is continuous over $[0,\overline{w}_\varepsilon')$. The proof of Proposition 6, with $\overline{w}=\overline{w}_\varepsilon'$ and $\overline{\overline{w}}=\overline{w}_\varepsilon$ yields a local SSE that achieves $(\overline{w}_\varepsilon,F_\varepsilon(\overline{w}_\varepsilon))$, with relational capital and incentive processes $\{w_t\}$, $\{F_\varepsilon(w_t)\}$. Given concavity of F_ε, $w_\varepsilon^*-\overline{w}_\varepsilon$ is small, as in the statement of the Theorem, establishing the proof.

Consider w such that $F''_\varepsilon(w) < -2\frac{r+\alpha}{\sigma_Y^2\varepsilon}$. Given quadratic costs, we have

$$(a(F_\varepsilon(w)) - c(a(F_\varepsilon(w))))' = \frac{1}{C}\left(1/2 - F_\varepsilon(w)\right) F'_\varepsilon(w).$$

Thus, differentiating (14), we get[40]

$$\begin{aligned} F''_\varepsilon(w) &= \frac{F'_\varepsilon(w)\left(\alpha+\gamma+[a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))]'\right)}{rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))} \qquad (37)\\ &= \frac{F'_\varepsilon(w)\left(\alpha+\gamma+\frac{1}{C}\left(1/2 - F_\varepsilon(w)\right) F'_\varepsilon(w)\right)}{rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))} \\ &\geq -\frac{F'^2_\varepsilon(w)}{2C|rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))|}, \quad \text{when } F'_\varepsilon(w) \leq 0 \\ &\geq -\frac{C_1 F'^2_\varepsilon(w)}{C(rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w))))}, \quad \text{when } F'_\varepsilon(w) \geq 0 \end{aligned}$$

where C_1 is the bound on F_ε from Lemma 3. For an appropriate $C_2 > 0$ this yields

$$\frac{F'^2_\varepsilon(w)}{|rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))|} \geq \frac{C_2}{\varepsilon}. \qquad (38)$$

On the other hand, equation (14) implies that

$$F'_\varepsilon(w)\left(rw - [a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))]\right) = (r+\alpha+\gamma)\, F_\varepsilon(w) \leq (r+\alpha+\gamma)\, C_1. \qquad (39)$$

Inequalities (38) and (39) imply that $|rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))| \leq C_3\varepsilon^{1/3}$, with $C_3 > 0$. Since $F_\varepsilon(w) \geq 1/2$, in the case when $F'_\varepsilon(w) \geq 0$ (so that the drift of relational capital is positive), whereas $F_\varepsilon(w) \geq \lim_{w\to w^*_\varepsilon} F_\varepsilon(w) = \underline{F}(w^*_\varepsilon) \geq C_4 > 0$, in the case when $F'_\varepsilon(w) \leq 0$ (the equality follows from the boundary condition (12)) equation (14) yields

$$|F'_\varepsilon(w)| = \frac{(r+\alpha+\gamma)\, F_\varepsilon(w)}{|rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))|} \geq C_5\varepsilon^{-1/3}. \qquad (40)$$

Since F_ε is concave and bounded in $[0, C_1]$, inequality (35) implies

$$\begin{aligned} w^*_\varepsilon - w &= O(\varepsilon^{1/3}), \quad \text{when } F'_\varepsilon < 0 \\ w &= O\left(\varepsilon^{1/3}\right). \quad \text{when } F'_\varepsilon > 0 \end{aligned}$$

It is enough now to show that the case $F'_\varepsilon(w) \geq 0$ is not possible. Note that since w is small and $rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))$ positive, we have $F_\varepsilon(w) \approx \overline{F}(0) = 1$. By

[40]Note that (14) implies $F'_\varepsilon(w) \times [rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))] \geq 0$.

differentiating (37),

$$
\begin{aligned}
F_\varepsilon'''(w) &= \left(\frac{F_\varepsilon'(w)}{rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))}\right)' \left(\alpha + \gamma + (a(F_\varepsilon(w)) - c(a(F_\varepsilon(w))))'\right) \qquad (41)\\
&+ \frac{F_\varepsilon'(w)}{rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))} \left(a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))\right)'' \\
&> \frac{F_\varepsilon'(w)}{rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))} \left(a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))\right)'' \\
&=_{sgn} \left(a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))\right)'',
\end{aligned}
$$

where the inequality follows from the fact that $F_\varepsilon''(w) < 0$ and

$$
\begin{aligned}
(rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w))))' &= r - \frac{1}{C}\left(1/2 - F_\varepsilon(w)\right) F_\varepsilon'(w) \\
&\approx r + \frac{1}{2C} F_\varepsilon'(w) > 0, \\
\alpha + \gamma + (a(F_\varepsilon(w)) - c(a(F_\varepsilon(w))))' &= \alpha + \gamma + \frac{1}{C}\left(1/2 - F_\varepsilon(w)\right) F_\varepsilon'(w) \\
&\approx \alpha + \gamma - \frac{1}{2C} F_\varepsilon'(w) < 0,
\end{aligned}
$$

when ε is small enough. Finally,

$$
\begin{aligned}
(a(F_\varepsilon(w)) - c(a(F_\varepsilon(w))))'' &= \left(\frac{1}{C}\left(1/2 - F_\varepsilon(w)\right) F_\varepsilon'(w)\right)' \qquad (42)\\
&=_{sgn} \left(1/2 - F_\varepsilon(w)\right) F_\varepsilon''(w) - \left(F_\varepsilon'(w)\right)^2 \\
&\approx -\frac{1}{2} F_\varepsilon''(w) - \left(F_\varepsilon'(w)\right)^2 \\
&\approx \frac{1}{4C} \frac{\left(F_\varepsilon'(w)\right)^2}{rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))} - \left(F_\varepsilon'(w)\right)^2 > 0,
\end{aligned}
$$

when ε is small enough, where the last line follows from (37). This establishes that $F_\varepsilon''(w^0) \leq -2\frac{r+\alpha}{\sigma_Y^2 \varepsilon}$ implies $F_\varepsilon'''(w^0) > 0$, and so the case $F_\varepsilon'(w^0) \geq 0$ is not possible. This establishes the proof of the Theorem.

Observe also that on $[0, \overline{w}_\varepsilon']$ we have the bounds

$$
\begin{aligned}
F_\varepsilon''(w) &\geq 2\frac{r+\alpha}{\sigma_Y^2 \varepsilon}, \qquad (43)\\
F_\varepsilon'(w) &\leq F_\varepsilon'(0) \leq \frac{1}{8rC} \times 2\frac{r+\alpha}{\sigma_Y^2 \varepsilon},
\end{aligned}
$$

where the second line follows from $w_{EF} = 1/8rC$ and $F_\varepsilon''(w) \geq 2\frac{r+\alpha}{\sigma_Y^2 \varepsilon}$ when $F_\varepsilon'(w) \geq 0$.

A.5 Proof of Theorem 3

Step 1. Fix $\varepsilon > 0$ and consider an $\varepsilon-$optimal local SSE $\{a_t, a_t\}$, together with the processes $\{w_t\}, \{F_t\}, \{I_t\}$ and $\{J_t\}$ that satisfy equations (9) (Proposition 2 and Theorem 2). In this step we show that as long as

$$J_t \leq \frac{C\left(r + 2\left(\alpha+\gamma\right)\right)}{8\left(r+\alpha\right)}, \quad \forall t \tag{44}$$

then, for an appropriate $X > 0$ and any deviating strategy $\{\widetilde{a}_t\}$, the relational capital at any time $\tau \geq 0$ to the deviating agent is bounded above by

$$\widetilde{w}_\tau(\widetilde{\mu}_\tau - \overline{\mu}_\tau, w_\tau) = w_\tau + \frac{F_\tau}{r+\alpha}(\widetilde{\mu}_\tau - \overline{\mu}_\tau) + X(\widetilde{\mu}_\tau - \overline{\mu}_\tau)^2. \tag{45}$$

In the formula, w_τ is the equilibrium level of relational capital, determined by (9), $\widetilde{\mu}_\tau$ are the correct beliefs, given strategies $\{\widetilde{a}_t\}$ and $\{a_t\}$, and $\overline{\mu}_\tau$ are the equilibrium beliefs, given that both strategies are $\{a_t\}$, both determined by (2). Consequently, using the bound with $\widetilde{\mu}_t = \overline{\mu}_t$, the step establishes that local SSE strategies are globally incentive compatible, as long as the bound (44) holds.

Fix a deviation strategy $\{\widetilde{a}_t\}$ and consider the process

$$v_\tau = \int_0^\tau e^{-rs}\left(\frac{\widetilde{a}_t + a_t}{2} - c(\widetilde{a}_t)\right)dt + e_\tau^{-r\tau}\widetilde{w}(\widetilde{\mu}_\tau - \overline{\mu}_\tau, w_\tau),$$

where, from (2), the wedge process $\{\widetilde{\mu}_t - \overline{\mu}_t\}$ follows

$$d\left(\widetilde{\mu}_t - \overline{\mu}_t\right) = \left(r+\alpha\right)\left(\widetilde{a}_t - a_t\right)dt - \left(\alpha+\gamma\right)\left(\widetilde{\mu}_t - \overline{\mu}_t\right)dt.$$

In order to establish that $\widetilde{w}_\tau$ bounds the relational capital under $\{\widetilde{a}_t\}$ and $\{a_t\}$, it is enough to show that the process $\{v_t\}$ has negative drift. We have

$$\begin{aligned} e^{-rt}dv_t = & \left(\frac{\widetilde{a}_t + a_t}{2} - c(\widetilde{a}_t)\right)dt - r\left(w_t + \frac{F_t}{r+\alpha}(\widetilde{\mu}_t - \overline{\mu}_t) + X(\widetilde{\mu}_t - \overline{\mu}_t)^2\right) \\ & + \left(rW_t - (a_t + c(a_t))\right)dt + I_t \times \left(dY_t - \overline{\mu}_t dt\right) \\ & + \frac{\widetilde{\mu}_t - \overline{\mu}_t}{r+\alpha}\left(\left(r+\alpha+\gamma\right)F_t - (r+\alpha)I_t dt + J_t \times \left(dY_t - \overline{\mu}_t dt\right)\right) \\ & + \left(\frac{F_t}{r+\alpha} + 2X(\widetilde{\mu}_t - \overline{\mu}_t)\right)\left(\left(r+\alpha\right)\left(\widetilde{a}_t - a_t\right)dt - (\alpha+\gamma)(\widetilde{\mu}_t - \overline{\mu}_t)dt\right). \end{aligned}$$

Given that the drift of $dY_t - \overline{\mu}_t dt$ is $(\widetilde{\mu}_t - \overline{\mu}_t)dt$, the drift of the $e^{-rt}dv_t$ process equals

$$\begin{aligned}
&\frac{\widetilde{a}_t - a_t}{2} + c(a_t) - c(\widetilde{a}_t) + F_t(\widetilde{a}_t - a_t) \\
&+ (\widetilde{\mu}_t - \overline{\mu}_t)^2 \left(\frac{J_t}{r+\alpha} - X\left(r + 2\left(\alpha+\gamma\right)\right) \right) + (\widetilde{\mu}_t - \overline{\mu}_t)(\widetilde{a}_t - a_t)2X\left(r+\alpha\right) \\
&\leq \frac{\widetilde{a}_t - a_t}{2} + c(a_t) - c(\widetilde{a}_t) + Ca_t(\widetilde{a}_t - a_t) \\
&+ (\widetilde{\mu}_t - \overline{\mu}_t)^2 \left(\frac{J_t}{r+\alpha} - X\left(r + 2\left(\alpha+\gamma\right)\right) \right) + (\widetilde{\mu}_t - \overline{\mu}_t)(\widetilde{a}_t - a_t)2X\left(r+\alpha\right) \\
&= -\frac{C}{2}\left(a_t - \widetilde{a}_t\right)^2 + (\widetilde{\mu}_t - \overline{\mu}_t)^2 \left(\frac{J_t}{r+\alpha} - X\left(r + 2\left(\alpha+\gamma\right)\right) \right) \\
&+ (\widetilde{\mu}_t - \overline{\mu}_t)(\widetilde{a}_t - a_t)2X\left(r+\alpha\right),
\end{aligned}$$

where we used that $c(a) = \frac{1}{2}a + \frac{C}{2}a^2$, and $F_t(\widetilde{a}_t - a_t) \leq Ca_t(\widetilde{a}_t - a_t)$, with equality in the case $a_t < A$.

Note that when the matrix

$$\begin{bmatrix} -\frac{C}{2} & X\left(r+\alpha\right) \\ X\left(r+\alpha\right) & \frac{J_t}{r+\alpha} - X\left(r+2\left(\alpha+\gamma\right)\right) \end{bmatrix}$$

has a positive determinant, then the trace is negative, and the matrix is negative semidefinite, guaranteing negative drift. Since

$$\begin{aligned}
&\max_X \left\{ -\frac{C}{2} \times \left(\frac{J_t}{r+\alpha} - X\left(r + 2\left(\alpha+\gamma\right)\right) \right) - X^2\left(r+\alpha\right)^2 \right\} \\
&= \frac{C}{2\left(r+\alpha\right)} \left(\frac{C\left(r+2\left(\alpha+\gamma\right)\right)}{8\left(r+\alpha\right)} - J_t \right),
\end{aligned}$$

it follows that, indeed, when J_t is bounded as in (44), then $\widetilde{w}_\tau$ defined in (45) bounds the relational capital, for X that maximizes the above expression.

Step 2. Fix $\varepsilon > 0$ and consider an $\varepsilon-$optimal local SSE $\{a_t, a_t\}$. In this step we show that when $C\sigma_Y$ is sufficiently large, then for any w_t the sensitivity J_t of relational incentives is bounded as in (44). Together with step 1, this will establish the proof of Theorem 3.

Recall from Proposition 6 and the discussion below that

$$J_t = J(w_t) = F'_\varepsilon(w) \times I^*_\varepsilon(w).$$

Let us bound $I_\varepsilon^*(w)$, in the case when $F_\varepsilon'(w) > 0$. (Since $I_\varepsilon^* \geq 0$, the bound (44) holds in the case when $F_\varepsilon'(w) \leq 0$.) Over the subset $S \subseteq [0, \overline{w}_\varepsilon)$ where $F_\varepsilon''(w) < -\frac{r+\alpha}{\sigma_Y^2 \varepsilon}$, we simply have $I_\varepsilon^*(w) = \varepsilon$. Over the complement $[0, \overline{w}_\varepsilon) \backslash S$, where $F_\varepsilon''(w) \geq -\frac{r+\alpha}{\sigma_Y^2 \varepsilon}$, we have,

$$
\begin{aligned}
I_\varepsilon^*(w) = -\frac{r+\alpha}{\sigma_Y^2 F_\varepsilon''(w)} &= \frac{2}{r+\alpha}\left\{ (r+\alpha+\gamma) F_\varepsilon(w) - F_\varepsilon'(w) \left(rw - (a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))) \right) \right\} \\
&\leq \frac{2}{r+\alpha}\left\{ \frac{(r+\alpha)^2}{256\sigma_Y^2 r^2 C^2} + r + \gamma + \alpha + \frac{r+\alpha}{4\sigma_Y^2 C r \varepsilon}\frac{1}{8C} \right\}, \\
&= \frac{r+\alpha}{128\sigma_Y^2 r^2 C^2} + \frac{2(r+\gamma+\alpha)}{r+\alpha} + \frac{1}{16\sigma_Y^2 C^2 r \varepsilon} =: I^{\#}
\end{aligned}
$$

where we use the bound (34) on F_ε, from Lemma 3, the bound $F_\varepsilon' \leq \frac{r+\alpha}{4\sigma_Y^2 C r \varepsilon}$ from (43), and the lower bound of $-(a_{EF} - c(a_{EF})) = -1/8C$ on the drift of relational capital.

Condition (44) thus boils down to

$$
J_t = F_\varepsilon'(w) \times I_\varepsilon^*(w) \leq \frac{r+\alpha}{4\sigma_Y^2 C r \varepsilon} \times (\varepsilon + I^{\#}) \leq \frac{C(r + 2(\alpha+\gamma))}{8(r+\alpha)},
$$

or,

$$
\varepsilon + \frac{r+\alpha}{128\sigma_Y^2 r^2 C^2} + \frac{2(r+\gamma+\alpha)}{r+\alpha} + \frac{1}{16\sigma_Y^2 C^2 r \varepsilon} \leq C^2 \frac{(r + 2(\alpha+\gamma))}{2(r+\alpha)^2} \sigma_Y^2 r \varepsilon, \tag{46}
$$

which is satisfied when $C\sigma_Y$ is large enough. This concludes the proof of the step, end of the first part of the theorem.

To verify the existence of non-trivial SSE, note that the policy function $I(w)$ in the proof of Proposition 3 equals zero at the extremes, and for any $w \in (0, \overline{w})$ satisfies

$$
I(w) \geq -\frac{r+\alpha}{\sigma_Y^2 D} \geq \frac{r+\alpha}{\sigma_Y^2} \frac{2}{2+r+\alpha+\gamma} \left(\frac{1}{16Cr} \right)^2 \geq \frac{1}{256\sigma_Y^2 C^2 r} =: \varepsilon. \tag{47}
$$

Proposition 3 establishes that for ε as in (47) the supremum w_ε^* of relational capitals achievable in $\varepsilon-$optimal local SSE is strictly above zero, as long as (24) is satisfied, which we reproduce below:

$$(r+\alpha+\gamma)\frac{2+r+\alpha+\gamma}{2}\left(\frac{r+\alpha+\gamma}{r}+\frac{1}{2}+2\left(2+r+\alpha+\gamma\right)\left(\frac{r}{r+\alpha+\gamma}\right)^2\right)\leq\frac{1}{512\sigma_Y^2C^2}. \tag{48}$$

Invoking (46) and (47), this local SSE is globally incentive compatible when

$$\frac{1}{256\sigma_Y^2C^2r}+\frac{r+\alpha}{128\sigma_Y^2r^2C^2}+\frac{2(r+\gamma+\alpha)}{r+\alpha}+\frac{256\sigma_Y^2C^2r}{16\sigma_Y^2C^2r}\leq\frac{(r+2\left(\alpha+\gamma\right))}{512\left(r+\alpha\right)^2},$$

or

$$\frac{r+\alpha+\gamma}{r}\frac{1}{256\sigma_Y^2C^2}+\left(\frac{r+\alpha+\gamma}{r}\right)^2\frac{1}{128\sigma_Y^2C^2}+2(r+\alpha+\gamma)+16(r+\alpha)\leq\frac{1}{512}. \tag{49}$$

For a given ratio $\frac{r+\alpha+\gamma}{r}$, inequalities (48) and (49) hold when, first, $C\sigma_Y$ is sufficiently large and, second, $r+\alpha+\gamma$ is sufficiently small. This concludes the proof of the theorem.

A.6 Proofs for Section 4

Proof of Proposition 4.

Part i) Suppose $\gamma=\sigma_\mu=0$. We show that the supremum w_ε^* of relational capitals achievable in the $\varepsilon-$optimal local SSE is increasing in σ_Y^{-1}, for every $\varepsilon>0$. Note that decreasing σ_Y changes equation (15) in Proposition 6 only by decreasing the last term. This means that if a pair of functions (F,I) satisfies the conditions of Proposition 6 for some interval $[\underline{w},\overline{w}]$ and a given σ_Y, then for any σ_Y' with $0<\sigma_Y'<\sigma_Y$ there is a function $\overline{I}\geq I$ such that the pair $\left(F,\overline{I}\right)$ satisfies the conditions of Proposition 6 for σ_Y'. Applying the result to the pair $(F_\varepsilon,I_\varepsilon^*)$ on the interval $[0,\overline{w}_\varepsilon]$ as in the proof of Theorem 2, for any $\varepsilon>0$, establishes the proof.

Part ii) Fix $\underline{w}>0$. Proof of part ii) of Proposition 3 establishes that a necessary condition for $w^*\geq\underline{w}$ is inequality (33), reproduced below:

$$1>32D^2C^2r^2\frac{\sigma_Y^2(r+\alpha+\gamma)}{(r+\alpha)^2}=\frac{64^3 2^6}{18}C^{10}r^8\left(\frac{\sigma_Y}{r+\alpha}\right)^6(r+\alpha+\gamma)^7\underline{w}^6. \tag{50}$$

Recall also that, when σ_Y is close to zero, γ is of order σ_Y^{-1} (see equation (3)). Substituting, the right hand side of (33) is of order σ_Y^{-1}, when σ_Y is close to zero. This establishes that

$w^* \leq \underline{w}$, when σ_Y is sufficiently small.

Part iii). As a preliminary step, we show that a symmetric strategy profile $\{a_t, a_t\}$ is an SSE with associated relational capital process $\{w_t\}$ if and only if there is an L^2 process $\{I_t\}$ such that

$$dw_t = (rw_t - (a_t - c(a_t)))\, dt + I_t \times (d\mu_t - [(r+\alpha)\, 2a_t - \alpha\mu_t]\, dt) + dM_t^w, \tag{51}$$

where $a_t = a((r+\alpha)\, I_t)$, and $\{M_t^w\}$ is a martingale orthogonal to $\{Y_t\}$, and the transversality condition $\mathbb{E}\left[e^{-rt}w_t\right] \rightarrow_{t\rightarrow\infty} 0$ holds.

The proof is identical to the first part of the proof of Proposition 2: since the process $\left\{\mu_t - \int_0^t \left[(r+\alpha)\, 2a_s - \alpha\mu_s\right] ds\right\}$, scaled by σ_μ, is a Brownian Motion, it follows from Proposition 3.4.14 in Karatzas [1991] that a process $\{w_t\}$ is the relational capital process associated with $\{a_t, a_t\}$, defined in (6), precisely when it can be represented as in (51), for some L^2 process $\{I_t\}$ and a martingale $\{M_t^w\}$ orthogonal to $\{\mu_t\}$.

As regards incentive compatibility, fix an alternative strategy $\{\widetilde{a}_t\}$ for player i and note that the relational capital satisfies

$$\begin{aligned}
&\mathbb{E}_\tau^{\{\widetilde{a}_t, a_t\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{\widetilde{a}_t + a_t}{2} - c(\widetilde{a}_t)\right) dt\right] \\
&= \mathbb{E}_\tau^{\{\widetilde{a}_t, a_t\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{\widetilde{a}_t + a_t}{2} - c(\widetilde{a}_t)\right) dt + w_\tau + \int_\tau^\infty d\left(e^{-rt}w_t\right)\right] \\
&= w_\tau + \mathbb{E}_\tau^{\{\widetilde{a}_t, a_t\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{\widetilde{a}_t + a_t}{2} - c(\widetilde{a}_t)\right) dt + \int_\tau^\infty e^{-rt}\left(dw_t - rw_t dt\right)\right] \\
&= w_\tau + \mathbb{E}_\tau^{\{\widetilde{a}_t, a_t\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{\widetilde{a}_t - a_t}{2} - c(\widetilde{a}_t) + c\left(a_t\right) + I_t\left(r+\alpha\right)\left(\widetilde{a}_t - a_t\right)\right) dt\right],
\end{aligned}$$

where the first equality follows from $\mathbb{E}_\tau^{\{\widetilde{a}_t^i, a_t^{-i}\}}\left[e^{-r(t-\tau)}w_t\right] \rightarrow 0$, as $t \rightarrow \infty$ (given that efforts are bounded), and the last one follows from $\mathbb{E}_\tau^{\{\widetilde{a}_t, a_t\}}\left[d\mu_t - \left[(r+\alpha)\, 2a_t - \alpha\mu_t\right] dt\right] = (r+\alpha)\, \mathbb{E}_\tau^{\{\widetilde{a}_t, a_t\}}\left[\widetilde{a}_t - a_t\right]$. Since continuation value and relational capital differ by a constant, it follows from this representation and convexity of costs that there exists no profitable deviating strategy for partner i if and only if her effort process satisfies $a_t = a((r+\alpha)\, I_t)$.

We are now ready to establish part iii) of the proposition. From representation (51) it follows that when $w_t \geq \varepsilon > 0$, then either the volatility satisfies $I_t \sigma_\mu \geq \delta > 0$, in order to incentivize a strictly positive, more efficient effort, or the drift satisfies $\mathbb{E}_\tau^{\{a_t^1, a_t^2\}}[dw_t] \geq \delta > 0$, to satisfy promise keeping (where δ depends on ε). It follows that if $w_0 > 0$ then the process $\{w_t\}$ escapes to infinity with positive probability, which, given bounded efforts, yields contradiction.

Proof of Proposition 5. Fix $\sigma_\mu > \sigma_\mu^\# \geq 0$; we show that, for any $\varepsilon > 0$, the corresponding suprema of relational capitals achievable in the $\varepsilon-$optimal local SSE satisfy $w_\varepsilon^{\#*} \geq w_\varepsilon^*$. The proof is very related to the proof of Proposition 4. One extra complication is that now, changing the noise of the fundamentals also affects the boundary conditions (16) in Proposition 6.

Specifically, note that decreasing σ_μ changes equation (15) in Proposition 6 only by decreasing γ in the first term. Let $\gamma^\# \leq \gamma$ be the two corresponding gain parameters, and let $\overline{w}_\varepsilon$ be the relational capital achievable in a $\varepsilon-$optimal local SSE with σ_μ, as in the proof of Theorem 2, together with a pair of functions $(F_\varepsilon, I_\varepsilon^*)$ defined on $[0, \overline{w}_\varepsilon]$. Let $\left(F_\varepsilon^\#, I_\varepsilon^{\#*}\right)$ extend the functions $(F_\varepsilon, I_\varepsilon^*)$ to the right by letting $F_\varepsilon^{\#''}(w) = F_\varepsilon^{\#''}(\overline{w}_\varepsilon)$ and $I_\varepsilon^{\#*}(w) = I_\varepsilon^{\#*}(\overline{w}_\varepsilon)$, for $w > \overline{w}_\varepsilon$, and let $\overline{w}_\varepsilon^\#$ be the first argument such that the boundary condition (16) is satisfied. The existence of such $\overline{w}_\varepsilon^\#$ follows from the fact that at $\overline{w}_\varepsilon$ condition (16) is violated, with the left-hand-side too small (due to $\gamma^\# \leq \gamma$), and when w increases and $F_\varepsilon^\#$ decreases and approaches from above the value $\underline{F}(w)$ at which the drift dies out, the left-hand-side is bounded away from zero, and the right-hand-side converges to zero.

It follows that for small $\sigma_\mu^\#$ and $\gamma^\#$, the pair $\left(F_\varepsilon^\#, I_\varepsilon^{\#*}\right)$ satisfies conditions of Proposition 6 with left inequality in the differential equation (15). Thus, there is a function $\overline{I} \geq I_\varepsilon^{\#*}$ such that the pair $\left(F_\varepsilon^\#, \overline{I}\right)$ satisfies the conditions of Proposition 6 on interval $[0, \overline{w}_\varepsilon^\#]$, with $\overline{w}_\varepsilon^\# > \overline{w}_\varepsilon$. This establishes the proof.

B Alternative Organizational Structures

Here we present simple models to capture the two alternative organizational structures presented in Section 4, and we provide formalizations for the relevant statements.

Career Concerns In some environments, although formal performance-contingent contracts are impossible to write, the workers may still be motivated to work hard to improve their reputation, that is by their own career concerns.

As in our main model, we are going to consider that production demands two agents' efforts (see equation 1). We can interpret the unknown quality of the joint venture, $\sigma_\mu B_t^\mu$, as the sum of two terms, capturing the quality of each worker. As in the main model, the market only observes a joint profit signal,

$$
\begin{aligned}
d\mu_t &= (r+\alpha)\left(a_t^1 + a_t^2\right)dt - \alpha\mu_t dt + \sigma_{\mu,1} dB_t^{\mu,1} + \sigma_{\mu,2} dB_t^{\mu,2}, \qquad (52)\\
dY_t &= \mu_t dt + \sigma_Y dB_t^Y.
\end{aligned}
$$

To simplify the analyzis, and in the spirit of Morrison and Wilhelm [2004] and of Bar-Isaac [2007], we focus on partnerships between an established partner, $\sigma_{\mu,1} = 0$, and a young partner, $\sigma_{\mu,2} > 0$. Finally, we capture the career concerns by assuming that each worker is paid according to the reputation of the joint venture, $\frac{\bar{\mu}}{2}$.

The following analysis and results are a direct consequence of Holmström [1999]. For completeness, we include the argument below. For player 1, exerting effort will not affect its reputation, hence we compute the marginal benefit of exerting effort for player 2. Increasing effort creates a wedge in the beliefs, with the productivity of the joint venture being above its reputation. In the future, the public news will be, on average, above expectations, pushing up player 2's reputation. The marginal benefit of effort is due to the benefit of increase in reputation.

Formally, the wedge created by an additional ε effort, at any instant, has magnitude $\varepsilon(r+\alpha)$.[41] The marginal benefit of such wedge is given by $\varepsilon(r+\alpha) \times \gamma/(r+\alpha+\gamma) \times 1/2(r+\alpha)$, where the second term is the expected discounted total effect of the wedge on

[41] More precisely, in the rest of this section, all the magnitudes that represent a wedge, marginal cost, or marginal benefit of effort exerted over an instant should be scaled by dt.

player's reputation, and the third term is the marginal value of an increase in reputation, given that each increase only slowly reverts to zero, and each worker captures only half of the reputation. Hence the marginal benefit of effort is given by

$$\text{marginal benefit of effort} = \frac{\gamma}{2(r+\alpha+\gamma)}.$$

It is clear that the marginal benefit of effort is increasing in γ, and thus improved monitoring facilitates the provision of effort, as claimed.

Contract. In the career concerns model we assumed that formal performance-contingent contracts were unavailable. We now relax this assumption. We consider an environment in which these contracts are available, although at a cost. Consider a principal that hires two workers for a joint task. To focus on the incentives provided by the contract and not by the relationship between the two employees, we consider that one of the positions is filled with a stream of workers hired by a short term contract, while the other is a long term employee. As in our main model, the productivity, μ_t, depends on the joint venture quality and on both workers efforts. We restrict attention to linear wages schemes that are stationary with respect to the unexpected news, $w_t = A + B * (dY_t - \overline{\mu}_t)$. We introduce contract imperfection in the model by assuming that the cost of a wage scheme is quadratic on the realized payment, with parameter δ; that is, for a transfer w_t, the principal incurs a cost of $w_t + \delta(w_t)^2$.

For a fixed contract, the marginal benefit of effort follows from the fact that exerting effort today generates a higher than expected fundamentals, and hence the future stream of news will be overall positive, giving the employee a positive stream of payments. Formally, the marginal benefit of effort is given by $\frac{B(r+\alpha)}{r+\alpha+\gamma}$, where the numerator follows from how additional effort creates a wedge between the actual productivity of the venture and the market expected productivity; and the denominator follows from the discount of such wedge. Note that for a fixed contract, the marginal benefit of effort decreases with γ, however the optimal contract is obviously not fixed.

The expected marginal cost of effort that the principal incurs in implementing a contract is given by $\delta B^2 \sigma_Y^2$. Hence for whatever marginal benefit of effort, mbe, that the principal wants to incentivize, the expected cost of such contract is $\delta\sigma_Y^2 mbe^2(\frac{r+\alpha+\gamma}{r+\alpha})^2$. Note that, for any marginal benefit of effort, the cost is increasing in the monitoring imperfection, giving the result claimed in the main text.

Formally, we have that the derivative is given by

$$\frac{\partial}{\partial\sigma_Y}(\frac{r+\alpha+\gamma}{r+\alpha})\sigma_Y = \frac{\partial}{\partial\sigma_Y}(\sigma_Y+\sigma_Y\frac{\gamma}{r+\alpha}) = 1+\frac{1}{r+\alpha}\frac{\partial}{\partial\sigma_Y}(\sigma_Y\gamma)$$
$$= 1+\frac{1}{r+\alpha}\frac{\partial}{\partial\sigma_Y}(\sqrt{\alpha^2\sigma_Y^2+\sigma_\mu^2}-\alpha\sigma_Y) = 1-\frac{\alpha}{r+\alpha}+\frac{\alpha}{r+\alpha}(\frac{\alpha\sigma}{\sqrt{\alpha^2\sigma_Y^2+\sigma_\mu^2}}) > \frac{r}{r+\alpha} > 0.$$

C General Model for Section 5

Suppose that an equilibrium is characterized by a system of equations governing the movement of the value G and the state variables $\theta \in\mathbb{R}^M$,

$$dG_t = [h^1(\theta_t,\mathbf{I}_t)\times G_t - h^2(\theta_t,\mathbf{I}_t)]dt + I^G dB_t, \tag{53}$$
$$d\theta_t = f(\theta_t,G_t,\mathbf{I}_t)dt + \sigma(\theta_t,G_t,\mathbf{I}_t)dB_t,$$

for some control processes $I^G \in \mathbb{R}$, $\mathbf{I}_t \in \mathbb{R}^N$ that are progressively measurable with respect to the Brownian Motion $\{B_t\}$, for Lipschitz continuous functions f, h^1, h^2, and σ, with h^1 positive and bounded away from zero, and for initial values θ_0, G_0.

Importantly, the only distinction between the value and the state variables in the system (53) is that value has drift linear in itself and has unrestricted volatility. Crucially, just as in the main model considered in the paper, value may affect the law of motion of the state variables. Unlike in the main model, the system may have multiple state variables, and the functions f, h^1, h^2, and σ are only assumed to be Lipschitz. In this formulation, we have the following analogues of Propositions 6 and 7. For any convex set S with a differentiable boundary and $s \in \partial S$ let $N(s)$ be the outward normal vector to ∂S an s.

Proposition 12 *Consider a compact convex subset $S \subset \mathbb{R}^N$ with a differentiable boundary ∂S such that $\theta_0 \in S$, together with a continuous function $\mathbf{I} : S \to \mathbb{R}^N$ and a C^2 strictly concave function $G : S \to R$ that satisfy the differential equation*

$$\begin{aligned} h^1(\theta, \mathbf{I}(\theta)) \times G(\theta) &= h^2(\theta, \mathbf{I}(\theta)) + \nabla G(\theta) \cdot f(\theta_t, G(\theta), \mathbf{I}(\theta)) \\ &+ \frac{1}{2}\sigma^T(\theta, G(\theta), \mathbf{I}(\theta)) \cdot \mathcal{H}(G) \cdot \sigma^T(\theta, G(\theta), \mathbf{I}(\theta)), \end{aligned} \tag{54}$$

where $\mathcal{H}(G)$ is the Hessian of G's partial second derivatives. Suppose also that $\partial S = \partial S^E \cup \partial S^R$, where each subset is measurable, and each boundary point $\theta \in \partial S^E$ together with $G(\theta)$ is achievable by an equilibrium, whereas each $\theta \in \partial S^R$ together with $G(\theta)$ satisfies

$$f(\theta_t, G(\theta), \mathbf{I}(\theta))^T \cdot N(\theta) \leq 0,$$
$$\sigma^T(\theta, G(\theta), \mathbf{I}(\theta))^T \cdot N(\theta) = 0.$$

Then, for every $\theta \in S$ there is an equilibrium achieving $(\theta, G(\theta))$.

Proof. Functions $G, \mathbf{I}$, defined on S, as well as f and σ are Lipschitz continuous. Let τ be the stopping time of a process $\{\theta_t\}$ reaching ∂S^E. The existence of a weak solution $\{\theta_t\}_{t \leq \tau}$ to the equations in (53), with $\mathbf{I}_t = \mathbf{I}(\theta_t)$ and $G_t = G(\theta_t)$ and $\theta_0 = \theta_0 \in S$ given, follows from, e.g., Karatzas [1991], Theorem 5.4.22. It follows from Ito's formula that the process $G_t = G(\theta_t)$ satisfies the first equation in (53). The pair of processes $\{\theta_t\}$ and $\{G_t\}$ is extended to $t > \tau$ by equating them with processes that characterize equilibria corresponding to $(\theta_\tau, G(\theta_\tau))$. This completes the proof. ■

Proposition 13 *Let $E(\theta)$ parametrize the supremum of G achievable in equilibrium, parametrized by $\theta \in S \subset \mathbb{R}^n$. For every $\lambda > 0$ there is $\delta > 0$ such that no concave solution G_λ of the differential equation*

$$\begin{aligned} G_\lambda(\theta) = \sup_{\mathbf{I}} \frac{1}{h^1(\theta, \mathbf{I})} \big\{ & h^2(\theta, \mathbf{I}) + \nabla G_\lambda(\theta) \cdot f(\theta_t, G_\lambda(\theta), \mathbf{I}) \\ &+ \frac{1}{2}\sigma^T(\theta, G_\lambda(\theta), \mathbf{I}) \cdot \mathcal{H}(G_\lambda) \cdot \sigma^T(\theta, G_\lambda(\theta), \mathbf{I}) \big\} + \lambda, \end{aligned} \tag{55}$$

on S, with $|\nabla G_\lambda| \leq 1/\lambda$, satisfies both of the following conditions:

$$i)\ G_\lambda(\theta) = E(\theta),\ \text{for all } \theta \in \partial S,$$

$$ii)\ 0 < E(\theta) - G_\lambda(\theta) \leq \delta.\ \text{for } \theta \in intS.$$

Proof. The proof by contradiction is completely analogous to the proof of Proposition 7. In particular, for the processes of $\{\theta_t\}$ and $\{G_t\}$ that characterize an equilibrium achieving θ_0 and G_0, with $\theta_0 \in intS$ and $0 < G_0 - G_\lambda(\theta) \leq \delta$, the distance function $D(\theta_t, G_t) = G_t - G_\lambda(\theta_t)$ satisfies, for appropriate process $\{\mathbf{I}_t\}$

$$\begin{aligned}
\frac{\mathbb{E}\left[dD(\theta_t, G_t)\right]}{dt} &= h^1(\theta_t, \mathbf{I}_t) \times G_t - h^2(\theta_t, \mathbf{I}_t) - \nabla G_\lambda(\theta_t) \cdot f(\theta_t, G_t, \mathbf{I}) \\
&- \frac{1}{2}\sigma^T(\theta_t, G_t, \mathbf{I}) \cdot \mathcal{H}(G_\lambda) \cdot \sigma^T(\theta_t, G_t, \mathbf{I}) \\
&\geq h^1(\theta_t, \mathbf{I}_t) \times G_t - h^2(\theta_t, \mathbf{I}_t) - \nabla G_\lambda(\theta_t) \cdot f(\theta_t, G_\lambda(\theta_t), \mathbf{I}) \\
&- \frac{1}{2}\sigma^T(\theta_t, G_\lambda(\theta_t), \mathbf{I}) \cdot \mathcal{H}(G_\lambda) \cdot \sigma^T(\theta_t, G_\lambda(\theta_t), \mathbf{I}) - \frac{\lambda}{2} \\
&\geq h^1(\theta_t, \mathbf{I}_t)\left(G_t - G_\lambda(\theta_t)\right) + \lambda - \frac{\lambda}{2} > h^1(\theta_t, \mathbf{I}_t) \times D(w_t, G_t),
\end{aligned}$$

with the inequalities following for small enough δ (from Lipschitz continuity of the parameter functions and the condition ii) in the proposition), and the definition of $G_\lambda(\theta)$. As in the proof of Proposition 7, the exponential expected growth rate of the distance D leads to violation of condition ii), establishing contradiction. ■

The two propositions jointly provide the HJB equation that characterizes the differentiable boundary of the supremum of G_t achievable in an equilibrium,

$$G(\theta) = \sup_{\mathbf{I}} \frac{1}{h^1(\theta, \mathbf{I})}\left\{h^2(\theta, \mathbf{I}) + \nabla G(\theta) \cdot f(\theta_t, G(\theta), \mathbf{I}) + \frac{1}{2}\sigma^T(\theta, G(\theta), \mathbf{I}) \cdot \mathcal{H}(G) \cdot \sigma^T(\theta, G(\theta), \mathbf{I})\right\}, \tag{56}$$

which is equation (54), in which the right-hand-side is pointwise maximized, and, at the same time, it is equation (55), with $\lambda = 0$. This is an analogue of equations (11), (15), and (17) in the main model in the paper. We note that that complete characterization of the equilibria, which would have to include the analysis of the boundary conditions, regularity properties of the boundary, as well as the solution of partial differential equation (56) is beyond the scope of this paper.

In the following section we consider special cases of the general model 53. In each case, G_t is the marginal benefit of increased fundamentals, and θ_t includes the level of fundamentals μ_t (or public estimate $\overline{\mu}_t$), continuation value W_t (or relational capital w_t) of one or more players, or marginal benefit of increased beliefs about fundamentals H_t. For each of the models we present the HJB differential equation (56).

C.1 Special Cases

The central features of the main model studied in this paper are the persistent effect of effort and the imperfect state monitoring. For tractability, our game and equilibria are symmetric, and the single state variable—the (expected) fundamentals $\overline{\mu}_t$—enters additively in the value of the partnership (see (52)). Here we present several extensions, together with the corresponding HJB equations. Those are special cases of the mathematical model (53) and the HJB equation (56).

Capital Accumulation. An important element for the provision of effort in teams, which we ignore in the main model, is that effort today may change the productivity of effort in the future. For instance, when developing a new product, early efforts to design a better product affect the productivity of later marketing efforts. To incorporate that, we allow the evolution of fundamentals μ_t to depend in a non-linear and non-separable way on the level of fundamentals μ_t and on effort a_t. We abstract from learning by assuming away the production noise, $\sigma_\mu = 0$, and, thus, $\overline{\mu}_t = \mu_t$ on the equilibrium path.[42] The fundamentals are not observed by the partners, $\sigma_Y > 0$. Specifically, the equations in (52) generalize to:

$$d\mu_t = g(\mu_t, a_t^1 + a_t^2)dt,$$
$$dY_t = \mu_t dt + \sigma_Y dB_t^Y,$$

where g is a differentiable function that is concave in the second argument.

[42] The non-linear evolution of fundamentals in this model would require non-linear learning, going beyond the Kalman-Bucy filter.

When g is nonlinear, one may not subtract relational capital w_t from continuation value W_t, nor separate relational incentives F_t from the total marginal benefit of increased fundamentals, G_t. Note that, in the main model, we have $w_t = W_t - \frac{\overline{\mu}_t}{2(r+\alpha)}$ and $F_t = (r+\alpha)G_t - 1/2$. Analogous to Proposition 2, W_t and G_t now follow:

$$\begin{aligned} dW_t &= \left(rW_t - \left(\frac{\mu_t}{2} - c(a_t)\right)\right) dt + I_t \times (dY_t - \mu_t dt) + dM_t^w, \\ dG_t &= \left(r - g_1(\mu_t, a_t^1 + a_t^2)\right) G_t dt - \left(\frac{1}{2} + I_t\right) dt + J_t \left(dY_t - \mu_t dt\right) + dM_t^G. \end{aligned} \tag{57}$$

The supremum of G across a local SSE depends on both the continuation value and the fundamentals, and the corresponding HJB equation (56) for the system (57) is

$$\begin{aligned} &(r - g_1(\mu, 2a)) \times G(W, \mu) = \\ &\max_I \left\{\frac{1}{2} + I + G_W \times (rW - (\mu/2 - c(a))) + G_\mu \times g(\mu, 2a) + \frac{G_{WW}}{2}\sigma_Y^2 I^2\right\}, \end{aligned} \tag{58}$$

where the locally optimal effort $a = a(G, \mu)$ satisfies $c'(a(G, \mu)) = g_2(\mu, 2a(G, \mu)) \times G$.

Oligopoly. Alternatively, we can generalize the model's payoff structure rather than changing the evolution of the fundamentals. This allows the model to speak to different economic settings, which we highlight with a simple model of an oligopoly. At any time, each firm i chooses to produce a quantity a_t^i, which adds up to the total stock of own goods produced, μ_t^i. At the same time, a fixed fraction α of total production is sold, with the mean price (inverse demand function) linearly decreasing in the quantity sold. Firms publicly observe only the price.[43] Formally, with π_t^i representing the cumulative profits of firm i until time t, we have[44]

$$\begin{aligned} d\mu_t^i &= (r+\alpha)a_t^i dt - \alpha\mu_t^i dt, \\ dY_t &= (p - \alpha(\mu_t^1 + \mu_t^2))dt + \sigma_Y dB_t^Y, \\ d\pi_t^i &= \alpha\mu_t^i dY_t - c(a_t^i)dt. \end{aligned}$$

The main difference between this model and the continuous time limit of the model

[43] Each firm also observes the fraction α of own goods sold, but this does not provide any information about the competitor.

[44] For simplicity, we keep the same scaling constants as in the body of the paper.

analyzed by Sannikov and Skrzypacz [2007] is that, here, the expected price at time t, dY_t, depends on the stock of goods produced in the past and gradually sold, rather than only on the current production (all of which is sold). The main difference with the main model in Section 2 of this paper is that, in a symmetric equilibrium with $a_t^1 = a_t^2, \mu_t^1 = \mu_t^2 =: \mu_t/2$, at any t, the flow revenue of firm i is $\frac{1}{2}\alpha\mu_t dY_t$, rather than $\frac{1}{2}dY_t$. (Additionally, the profits would differ in asymmetric equilibria.)

The continuation value function W_t and the marginal benefit of increased own stock of goods G_t in a symmetric equilibrium satisfy

$$\begin{aligned} dW_t &= \left(rW_t - \left(\frac{1}{2}\alpha\mu_t(p-\alpha\mu_t) - c(a_t)\right)\right)dt + I_t \times (dY_t - (p-\alpha\mu_t)dt) + dM_t^W, \\ dG_t &= (r+\alpha)\,G_t dt + (-\alpha I_t + \alpha(p-\alpha\mu_t))dt + J_t \times (dY_t - (p-\alpha\mu_t)dt) + dM_t^G, \end{aligned} \tag{59}$$

and the corresponding HJB equation (56) for the system (59) is:

$$\begin{aligned} (r+\alpha)\times G(W,\mu) = \max_I \Bigg\{ &-\alpha I_t + \alpha(p-\alpha\mu_t) + \\ G_W \times \left(rW_t - \left(\frac{1}{2}\alpha\mu_t(p-\alpha\mu_t) - c(a_t)\right)\right) &+ G_\mu \times (2(r+\alpha)a - \alpha\mu) + \frac{G_{WW}}{2}\sigma_Y^2 I^2 \Bigg\}, \end{aligned}$$

where $a = a(G)$ is the locally optimal action that satisfies $c'(a(G)) = (r+\alpha)\times G$.

Asymmetric game and PPE. In joint ventures, partnerships, and teams, the relationship may be asymmetric: an individual may be more or less productive than another; partners may not share rewards equally; and partners may have different costs for exerting effort. To consider a general team production, we would need to extend our framework to include asymmetric PPE. Towards that goal, consider a partnership model with $N \geq 2$ players, with

$$\begin{aligned} d\mu_t &= -\alpha\mu_t dt + g(a_t^1, a_t^2, ..., a_t^N)dt, \\ dY_t &= \mu_t dt + \sigma_Y dB_t^Y, \\ d\pi_t^i &= \beta^i dY_t - c^i(a_t^i)dt, \end{aligned}$$

where g is differentiable and concave in each argument; each individual $i = 1, 2, ..., N$ has a cost of effort c^i as in the main model; and each i receives a fixed share of the profit β^i.

The relational capitals w^i and the marginal benefit of increased fundamentals G^i follow

$$dw_t^i = \left(rw_t^i - \left(\frac{g(a_t^1, a_t^2, ..., a_t^N)}{r+\alpha} - c^i(a_t^i) \right) \right) dt + I_t^i \times (dY_t - \mu_t dt) + dM_t^{w^i}, \qquad (60)$$
$$dG_t^i = (r+\alpha) G_t^i dt - \left(\beta^i + I_t^i \right) dt + J_t^i \times (dY_t - \mu_t dt) + dM_t^{G^i}.$$

For a model with two players, the HJB equation (56) that characterizes the supremum G^1 of the marginal benefit of increased fundamentals for player 1 as a function of two relational capitals and marginal benefit G^2 of increased fundamentals for player 2, as in (60) is

$$(r+\alpha) \times G^1(w^1, w^2, G^2) = \qquad (61)$$
$$\max_{I^1, I^2, J^2} \left\{ \begin{array}{l} \beta^1 + I^1 + G^1_{w^1} \times \left(rw^1 - \frac{g(a^1,a^2)}{r+\alpha} + c^1(a^1) \right) + G^1_{w^2} \times (rw^2 - \frac{g(a^1,a^2)}{r+\alpha} + c^2(a^2)) \\ +G^1_{G^2} \left((r+\alpha) G^2 - \left(\beta^2 + I^2 \right) \right) + \beta^1 \left[\begin{array}{ccc} I^1 & I^2 & J^2 \end{array} \right] H(G^1) \left[\begin{array}{ccc} I^1 & I^2 & J^2 \end{array} \right]^T \end{array} \right\},$$

where $H(G^1)$ is the Hessian of partial second derivatives of G^1, and the locally optimal efforts $a^i = a^i(G^1, G^2)$ satisfy $c^{i\prime}(a^i(G^1, G^2)) = g_i\left(a^1\left(G^1, G^2\right), a^2\left(G^1, G^2\right)\right) \times G^i$.

Selling the partnership and a nonstationary model. An alternative interpretation of a partnership unraveling is that the partners sell it. If the venture's market value is the value of its fundamentals, partners part with it when relational capital and, thus, their value added dries up. Realistically, the market value of a venture may well exceed its fundamentals' value. In this case, if partners cannot commit not to sell the partnership, the scope for incentives diminishes.

For example, when the market offers a fixed markup m above the fundamentals' value, the relational capital may not decrease below this level. Formally, the left boundary condition in equation (11) changes, so that $w \geq m$, in any local SSE.

When the market's markup is a fixed fraction of the fundamentals' value—say, $m = 2\overline{\mu}$—the boundary condition includes both the fundamentals' value and relational capital, so that $w \geq 2\overline{\mu}$. This requires adding the fundamentals' value as the second state variable in the equation, as in (58), even with the additive formulation of Section 2.

Finally, the partnership might unravel exogenously, becoming less efficient in producing output over time. More broadly, we may consider a nonstationary model of partnership, in which all the parameters—depreciation of capital, α_t; individual cost of effort, c_t; marginal effect of effort on fundamentals ($r + \alpha$ in the main model, see 52), K_t; production and observation noises, $\sigma_{\mu t}$ and σ_{Yt}; etc—change deterministically over time. Formally, this would require introducing time as an additional state variable. The processes described in Proposition 2 change so that all parameters are indexed by time, whereas the HJB equation for relational incentives becomes

$$(r + \alpha_t + \gamma_t)F(w,t) = \\ \max_I \left\{ K_t \times I + F_w(w,t)\left(rw - (a - c_t(a))\right) + F_t(w,t) + \frac{F_{ww}(w,t)\sigma_{Yt}^2}{2} I^2 \right\},$$

with effort $a = a(F)$ defined in (8).[45]

[45] In case of the nonstationary learning, it follows from the Kalman-Bucy filtering equations (see Liptser and Shiryaev [2013]) that the equilibrium posterior variance about the fundamentals, σ_t^2, and the gain parameter, γ_t, satisfy $\sigma_t^{2\prime} = -2\alpha_t\sigma_t^2 + \sigma_{\mu t}^2 - \gamma_t^2\sigma_{Yt}^2$ and $\gamma_t = \sigma_t^2/\sigma_{Yt}^2$, with an exogenous prior variance σ_0^2.