Partnership with Persistence

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Abstract

We study a continuous-time model of partnership with persistence and imperfect state monitoring. Partners exert private efforts to shape the stock of fundamentals, which drives the profits of the partnership, and the profits are the only public signal. The optimal strongly symmetric equilibria are characterized by a novel differential equation that describes the supremum of equilibrium incentives for any level of relational capital. Under (almost) perfect monitoring of fundamentals, the only equilibria are (approximately) stationary Markov. Imperfect monitoring helps sustain relational incentives and increases the partnership's value by extending the relevant time horizon for incentive provision. The results are consistent with the predominance of partnerships and relational incentives in environments where effort has long-term and qualitative impact and in which progress is hard to measure.

Keywords: partnership, dynamic games, continuous time, relational incentives

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1 Introduction

Partnerships are among the main forms of organizing economic activity. Characterized by joint ownership, partnerships are common among individuals and businesses and constitute one of the dominant forms of structuring a firm—along with corporations and sole proprietorships. Furthermore, partnerships embody the incentive problem of motivating members to exert private effort and contribute to the common good, which is common to many organizations.[1]

The ongoing, dynamic nature of joint ownership complicates the incentive problem in a partnership. As an example, consider a start-up. On a daily basis, each partner devotes effort to improving the venture's fundamentals: upgrading the quality of the product; broadening the customer base; facilitating access to external capital; improving the internal organization; and more. Each of these fundamentals evolves over time, shaped by the partners' current and past efforts, and by the circumstances. Moreover, none of them needs to be directly observed by the partners, who see only how the fundamentals are gradually reflected in the shared profits, customer reviews, or internal audits. At the same time, the ongoing nature of joint ownership offers unique opportunities to solve the incentive problem: it fosters relational incentives. A partner has incentives to work hard not only to boost profits but also to boost observable outcomes, morale, and, ultimately, to increase the future effort choices of the partners.

In this paper, we analyze incentives in a continuous-time model of partnership, with a persistent, stochastic state—the fundamentals—that is driven by the partners' actions and is imperfectly monitored by them. We provide a novel ordinary differential equation that characterizes the upper boundary of equilibrium incentives and the supremum of partnership values, in a strongly symmetric equilibrium. Our main substantive contribution is to identify a new channel through which imperfect monitoring of a persistent,

[1] According to the IRS data, in 2015 partnerships made up over 10% of all U.S. businesses, and accounted for over 25% of total net business income; see `https://www.irs.gov/statistics/soi-tax-stats-integrated-business-data`, Table 1. More broadly, teamwork, which shares the central feature of collective rewards and free-riding, was utilized in close to 80% of U.S. businesses at the turn of the century; see Lazear and Shaw (2007).

stochastic state may increase equilibrium payoffs. If persistent fundamentals are perfectly monitored, the partnership cannot provide relational rewards at the highest equilibrium continuation value—i.e. the bliss point—and the relational incentives unravel, or nearly so when fundamentals are monitored closely. Imperfect monitoring benefits partners in that it allows future relational rewards to motivate today's effort, alleviating the incentive problems at the bliss point. The results are consistent with the predominance of partnerships and relational incentives in environments in which effort has long-term and qualitative impact and in which progress is hard to measure.

In our continuous-time model, at any point in time, two partners privately exert costly effort and evenly split the profits of their venture. Persistent fundamentals evolve stochastically, driven by the sum of efforts, and, in turn, equal the expected profit flow. In the model, neither efforts nor fundamentals are observable and profits, which follow a Brownian diffusion, are the partners' only publicly available information. Our minimal monitoring structure does not allow the signals to separately identify each partner's effort (Fudenberg, Levine, and Maskin (1994)) and, consequently, we focus on strongly symmetric equilibria (SSE).

A partner's effort increases the fundamentals of the partnership and, thus, profits in the future. This benefits her in two distinct ways. First, she benefits directly by capturing half of the increased profits. In our model, those *Markov incentives* are constant, resulting in a unique, stationary Markov equilibrium (Proposition 1). Second, the increased profits affect the partners' effort decisions, such as when partners coordinate on relatively efficient (inefficient) efforts after surprisingly high (low) profit realizations, indicative of high (low) past efforts. These *relational incentives* constitute the key focus of this paper.

Our environment combines persistent effects of effort and imperfect state monitoring. Together, these two elements affect the incentives in a partnership by lengthening the time horizon for incentive provision. If the current profits depend only on current efforts (i.e., in a standard imperfectly public monitoring repeated game) or if fundamentals are perfectly monitored (i.e., in a standard stochastic game), then the rewards must be provided instantaneously. This is because an increased effort in a period brings about unexpectedly good news (higher than expected profits in the repeated game and fundamentals in the stochastic game) only in that same period. Our main model lies outside of those two limiting environments, and hence signals indicating an increased effort today are spread over time. As a consequence, rewards awarded for unexpectedly high profits in the future provide incentives to exert effort today. This change has a dramatic impact on the provision of relational incentives, as we discuss below.

We show that with persistent effect of effort, poorer monitoring may improve relational incentives and, thus, benefit the partnership. The first part of the argument is specific to relational incentives: relational rewards must take the form of a promise of an improved future relationship. However, at the highest equilibrium continuation value for the partners, i.e. when the relationship is at its bliss point, no relational rewards are available. The second part of the argument is that over a short interval of time, the information in profits about the efforts is imprecise. Hence, incentives cannot be provided solely with punishments, as this would be used mistakenly too often. Taken together, when the time horizon for incentive provision is short, effort cannot be incentivized near the bliss point of the relationship and the construction of relational incentives essentially unravels. We show that, in contrast, with poorer monitoring, nontrivial relational incentives are possible. In the absence of immediate rewards, partners are incentivized to work at the bliss point by future rewards, accruing once the continuation value of the relationship drifts down. While the impossibility of relational incentives in continuous-time partnerships and cartels has been attributed to the high frequency of actions (see Sannikov and Skrzypacz (2007, 2010)), our results indicate that it stems from the assumption of perfect state monitoring.

The effects of relational incentives on effort dynamics and comparative statics of partnership value go beyond the benefit of poor monitoring. First, we show that following good outcomes (higher profits), partners increase their efforts when the value of the relationship is low and close to unraveling, but decrease efforts when the value is high, and good outcomes push it closer to the bliss point with a vanishing flow of relational incentives ("rallying and coasting"). Second, the impact of outcomes on effort is decoupled from their impact on expected fundamentals. As a consequence, an established partnership may unravel following a short spat of bad outcomes, with hardly any effect on its expected fundamentals ("Beatles' break-up".) Third, we show that besides the monitoring precision, the persistence of effort and uncertainty about the partnership's quality are key factors shaping the time window for the provision of relational incentives and the partnership value. Relational incentives vanish in partnerships with transient effects of effort, as when the partners manage the venture following well-established routines, rather than making strategic decisions for the future. Moreover, less uncertainty about the quality of the partnership is always beneficial to the partners. Overall, the effects that the informational environment has on relational incentives contrast with the opposite effects it would have on career concerns and reputational incentives. Partnerships thus have an edge in environments that favor established ventures, and in which the effects of effort are long-lived and hard to measure or quantify (such as in the professional sector, Levin and Tadelis (2005)).

The possibility of nontrivial relational incentives requires a novel method to characterize the optimal incentive provision in an environment with persistence. The method we propose is based on characterizing the upper boundary of relational incentives achievable in a local SSE, under local incentive constraints, as a function of the expected value of future efforts (relational capital, an equivalent of continuation value in an i.i.d. setting). Theorem 1 shows that the upper boundary of incentives satisfies a version of the HJB ordinary differential equation and provides the appropriate boundary conditions. Theorem 2 shows, roughly, that a modified boundary is self-generating (as in (Abreu, Pearce, and Stacchetti, 1990)) and defines a near-optimal local SSE.

One difficulty in our approach, with incentives as the value function, is that the law of motion of the state variable (relational capital) depends, via effort chosen, on the level of the value function. While the dependence is not allowed in standard stochastic control, we verify that the characterization of the boundary in the form of the HJB equation is valid.[2]

[2] We are aware of abusing the terminology. Formally, the optimality equation is not an HJB equation, as the state variable depends on the objective function. However, the optimality equation has the exact form of the HJB equation, with incentives F equal to the point-wise maximum, over all policies, of the expected flow of incentives, plus the stochastic differential operator applied to function F. We believe

Another difficulty is familiar in the literature; in Theorem 3, we provide conditions on the primitives so that the constructed strategies are not only locally, but globally incentive-compatible.

1.1 Related Literature

This paper belongs to the literature on free-riding in groups, in dynamic environments.[3] We consider a canonical partnership model in the spirit of Radner (1985) and Radner, Myerson, and Maskin (1986), who first showed the inefficiency of equilibria in partnerships. Our main contribution is to provide a tractable solution of a partnership model that includes persistence and imperfect monitoring of the state, and to show that it can sustain nontrivial relational incentives while providing rich equilibrium dynamics (in contrast to 'bang-bang" results, see Abreu, Pearce, and Stacchetti (1986)) and comparative statics. Below we relate our construction of relational incentives to the impossibility results in the literature; relate it to the provision of incentives in the literature on models with hidden states; discuss how our method departs from standard stochastic control and relates to previous work in dynamic contracts and continuous-time games; and discuss the literature on partnerships on broader informational contexts.

Our results provide a novel rationale for the impossibility of relational incentives found in the literature. Abreu, Milgrom, and Pearce (1991) and Sannikov and Skrzypacz (2007, 2010) show how frequent interactions may have a detrimental effect on the provision of incentives. In particular, the model of partnership or collusion in Sannikov and Skrzypacz (2007) has either no persistence or a perfectly monitored state and the impossibility is attributed to the fact that, in a discrete-time approximation of a Brownian motion, news arrive and actions are taken frequently.[4] We show that the impossibility is not inherent to continuous-time modeling and is a consequence of the monitoring structure.

that our results are best viewed as extending stochastic control methods to allow for such dependency.

[3]See Olson (1971), Alchian and Demsetz (1972), Holmstrom (1982), as well as Legros and Matthews (1993) and Winter (2004) for seminal contributions in static settings.

[4]See Faingold and Sannikov (2011) and Bohren (2018) for related impossibility results with one long-lived player in a competitive market setting. See, also, Sadzik and Stacchetti (2015) for the discrete-time approximation of the Brownian Principal-Agent, rather than the partnership model.

More broadly, while a number of papers establish that better information may have a detrimental effect on the provision of incentives, our contribution is to identify a novel channel for the negative effect of better monitoring on relational incentives, i.e. the shortening of the time horizon for incentives.[5]

The provision of incentives in a partnership of unknown quality relates to the literatures on career concerns (see Holmström (1999) and Cisternas (2017)) and on experimentation in teams (see Bolton and Harris (1999), Georgiadis (2014), and Cetemen, Hwang, and Kaya (2020)). In career concerns models, equilibrium play depends only on beliefs about an exogenous (Markov) state;[6] the literature on experimentation in teams studies the effects of payoff or information externalities on incentives and focuses on Markov equilibria as well. Our paper is complementary: It has a production technology that is independent of history, with constant Markov incentives. Our primary focus is, thus, to characterize the provision of relational incentives in equilibrium, going beyond Markov equilibria. In a similar vein, Hörner, Klein, and Rady (2022) investigate relational incentives however their focus is on the experimentation in teams model.

While persistence is a pervasive feature of the real world, the literature on dynamic contracting has long recognized how it complicates the accounting of incentives or verifying global incentive compatibility (see Jarque (2010), Williams (2011), Prat and Jovanovic (2014), Sannikov (2014), Prat (2015), DeMarzo and Sannikov (2016), and He et al. (2017) for Brownian models like ours). Our solution method departs from the techniques used in this literature as, in our game setting, incentives are not treated as a state variable but as a maximized objective. Our HJB characterization, which allows for the evolution of the state variables to depend on the level of the value function, is related to the HJB characterization in Sannikov (2007), which allows the dependence on the slope of the

[5]In a non-relational environment, in a model of non-contractible performance measures, better monitoring can exacerbate the principal's exploitation motive (see Zhu (2023)) while in Cetemen, Hwang, and Kaya (2020), limited feedback may help partners by mitigating the ratchet effect. Similarly, in a linear Gaussian rating model of Hörner and Lambert (2021), Bonatti and Cisternas (2020) show that putting relatively too much weight on old signals about a customer may mitigate the ratchet effect and benefit the firm (see also Ball (2022)).

[6]Specifically, Cisternas (2017) provides a differential equation also for the stock of incentives, just as in this paper, but in a differentiable Markov equilibrium, as a function of public beliefs about the state.

value function, in a repeated game setting. With an eye on extending the analysis to stochastic games with imperfect monitoring, Faingold and Sannikov (2020) and Bernard (2023) focus on games which, unlike a partnership studied here, satisfy identifiability conditions, allowing players to identify and punish possible individual deviators. As an additional technical contribution, we provide conditions on the primitives of the model for the global incentive compatibility, guaranteeing that the solution of the relaxed problem is fully incentive-compatible (see Williams (2011), Edmans et al. (2012), Cisternas (2017), and Di Tella and Sannikov (2021) for related results).[7]

Finally, we interpret our results as providing a rationale for the prevalence of partnerships in industries with poor monitoring of the venture's progress. In particular, as documented by Von Nordenflycht (2010), "opaque" quality is a key characteristic of the knowledge-intensive environment of the professional sector, where partnerships are prevalent.[8] Our results are related to Levin and Tadelis (2005), who rely on partnership's comparative advantage in industries where employee quality is hard to evaluate, and to Morrison and Wilhelm (2004), who focus on partnership's impact on fostering mentorship relations. In contrast to these, we focus on the incentive provision for partnerships given imperfect monitoring of its quality.

2 Framework

2.1 Model

Two partners, who are risk-neutral and discount the future at a rate $r > 0$, play the following infinite horizon game: At every moment in time, $t \geq 0$, each partner i chooses effort a_t^i from an interval $[0, A]$.[9] Time t total effort contributes to the stock of funda-

[7]Zhu (2013) and Grochulski and Zhang (2023) face similar problems to us in constructing contracts based on "Sticky Brownian Motion", such as our relational capital with volatility vanishing discontinuously at the boundary (see Lemma 2); see also Engelbert and Peskir (2014) for the related existence results.

[8]See Empson (2001) and Broschak (2004) for further references.

[9]The upper bound on the effort is used to guarantee boundedness of continuation value in Propositions 1 and 3, part iii). In all simulations, as well as in Theorem 2, the bound A is large enough so that equilibrium efforts are interior.

mentals of the partnership, μ_t, which depreciates over time at a constant rate $\alpha > 0$. At any point in time, the stock of fundamentals is the mean of the partnership flow profits dY_t,

$$d\mu_t = (r+\alpha)\,(a_t^1 + a_t^2)dt - \alpha\mu_t dt + \sigma_\mu dB_t^\mu, \tag{1}$$
$$dY_t = \mu_t dt + \sigma_Y dB_t^Y,$$

where $\{B_t^\mu\}$ and $\{B_t^Y\}$ are two independent Brownian Motions, $\sigma_\mu, \sigma_Y \geq 0$.[10] The multiplicative constant, $r+\alpha$, in (1) normalizes the total discounted marginal benefit of effort to one, regardless of the depreciation rate of the fundamentals or of the discount rate.[11] Finally, profits are the only publicly observable signal.

Exerting effort a entails a private flow cost $c\,(a)$, where $c(\cdot)$ is a twice differentiable, strictly convex cost of effort function. Throughout the paper, we normalize $c(0) = 0$ and $c'(0) = \frac{1}{2}$ (see Proposition 1) and, in Sections 3.1 and 4, we further restrict the cost function to be quadratic. Finally, at each point in time, partners split the profits evenly. Thus, given effort choices of both partners, a player's continuation payoffs are given by

$$W_\tau^i = \mathbb{E}_\tau^{\{a_t^1, a_t^2\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{\mu_t}{2} - c(a_t^i)\right)dt\right].$$

Aside from the continuous-time modelling, our partnership game has three features that extend both the repeated game and the stochastic game framework: effort has persistent effect, state is imperfectly monitored, and partners learn about the fundamentals. Specifically, fundamentals—which are the state variable in the game—change only gradually over time driven by the efforts of the partners. Persistence of fundamentals implies that actions have a persistent effect: fundamentals, and, thus, also the profit today depends on the integral of efforts at any previous time,

$$\mu_\tau = e^{-\alpha\tau}\mu_0 + \int_0^\tau e^{-\alpha(\tau-t)}(r+\alpha)\left(a_t^1 + a_t^2\right)dt + \sigma_\mu\int_0^\tau e^{-\alpha(\tau-t)}dB_\tau^\mu. \tag{2}$$

[10] Unless specified explicitly, all processes in this paper are indexed by $t \geq 0$. We require that either σ_Y or σ_μ is strictly positive, to avoid the familiar complications in defining a continuous-time strategy in a game with perfect monitoring.

[11] The normalization simplifies the statements of the results. We verify in the proof of Proposition 2 that the comparative statics results with respect to α and r continue to hold without the normalization.

This contrasts with the repeated game framework, in which today's fundamentals and profits depend only on today's efforts. Secondly, unlike in a stochastic game, fundamentals need not be observed by the partners, who observe noisy profit signals. Together, the two features imply that in contrast to the repeated or stochastic game version of the model analyzed in Sannikov and Skrzypacz (2007), all future profits are signals of current efforts (see Lemma 1 and Section 4.2). Thirdly, fundamentals need not be determined by the efforts, and change stochastically. Alternatively, fundamentals are a sum of two terms: one that depends entirely on the past efforts of the partners, and the other that is purely stochastic and reflects an unknown quality of the partnership, in the spirit of Holmström (1999). Consequently, in equilibrium partners do not know and keep on learning about the fundamentals.

The three features are parametrized in the model by α, σ_Y, σ_μ.[12] Their impact on the incentive provision in a partnership is one of the central themes of the paper, and we discuss it at length in the following sections.

Public Beliefs Let $\overline{\mu}_\tau = \mathbb{E}_\tau^{\{a_t^1, a_t^2\}}[\mu_\tau]$ denote the public expected level of fundamentals at time τ, given efforts $\{a_t^1, a_t^2\}$. An application of the Kalman-Bucy filter yields that $\overline{\mu}_t$ follows

$$d\overline{\mu}_t = (r+\alpha)\,(a_t^1 + a_t^2)dt - \alpha\overline{\mu}_t dt + \gamma_t[dY_t - \overline{\mu}_t dt], \tag{3}$$

for an appropriate gain parameter γ_t, $dY_t = \overline{\mu}_t dt + \sigma_Y dB_t$, and a Brownian Motion $\{B_t\}$. We assume that, initially, partners believe that μ_0 is Normally distributed with steady-state variance σ^2. This implies that both the posterior estimate variance σ_t^2 and the gain parameter γ_t remain constant throughout the game and equal (see Liptser and Shiryaev (2013))

$$\gamma = \sqrt{\alpha^2 + \frac{\sigma_\mu^2}{\sigma_Y^2}} - \alpha, \text{ and } \sigma^2 = \gamma \times \sigma_Y^2. \tag{4}$$

[12]The imperfect public monitoring repeated game is obtained by having $\mu_\tau = a_\tau^1 + a_\tau^2$. In our model, this can be approximated when $\sigma_\mu = 0$ and the mean-reversion parameter α converges to infinity, and thus the integral in Equation (2) converges pointwise to the sum of efforts at time t; see, among others, Sannikov (2014).

2.2 Equilibrium

A player's (pure, public) strategy $\{a_t^i\}$ is a progressively measurable process that depends on the public information $\{Y_t\}$ and allows for public randomization. A pair of public strategies, $\{a_t^1, a_t^2\}$, is a *perfect public equilibrium (PPE)* if, for each partner i at any time $\tau \geq 0$,

$$\mathbb{E}_\tau^{\{a_t^i, a_t^{-i}\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{\mu_t}{2} - c(a_t^i)\right)dt\right] \geq \mathbb{E}_\tau^{\{\widetilde{a}_t^i, a_t^{-i}\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{\mu_t}{2} - c(\widetilde{a}_t^i)\right)dt\right], \quad (5)$$

following any history, for any possible alternative strategy $\{\widetilde{a}_t^i\}$.[13]

Markov Equilibria In a *Markov equilibrium*, play depends on the past history via the minimal set of payoff-relevant states.[14] Our model, however, is linear: the evolution of the state μ is linear in the sum of efforts and the evolution of expected profits is linear in the state. Given no effect of past efforts on the profitability of the current effort, the model admits a unique Markov equilibrium, which is stationary.

Proposition 1 *A pair of constant strategies $\{a_t, a_t\}$ in which partners never exert effort, $a_t = 0$, for every $t \geq 0$, constitutes a PPE. It is the unique stationary PPE, and so it is the unique Markov equilibrium.*

The argument is as follows. Exploiting the linear structure of the model, we may rewrite the continuation payoffs as

$$\begin{aligned} W_\tau^i &= \mathbb{E}_\tau^{\{a_t^1, a_t^2\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{\mu_\tau}{2}e^{-\alpha(t-\tau)} + \int_\tau^t (\alpha + r)\frac{a_s^1 + a_s^2}{2}e^{-\alpha(t-s)}ds - c(a_t^i)\right)dt\right] \quad (6) \\ &= \frac{1}{2(r+\alpha)}\mathbb{E}_\tau^{\{a_t^1, a_t^2\}}[\mu_\tau] + \mathbb{E}_\tau^{\{a_t^1, a_t^2\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{a_t^1 + a_t^2}{2} - c(a_t^i)\right)dt\right]. \end{aligned}$$

The first term in the last line of (6) captures the expected value of "inherited" (expected) fundamentals to a partner. Even if at some time τ partners stop exerting effort,

[13] We note that a public strategy does not restrict a partner to condition only on the public expectation $\overline{\mu}_\tau$, or to revert to the equilibrium path immediately after a deviation; say, a strategy that lets a partner shirk first, and then play depending on own estimate of the fundamentals depends only on clock time and public signals.

[14] See Mailath et al. (2006), Definition 5.6.1 on page 191, for a formal definition of Markov equilibrium.

the fundamentals will only slowly revert to zero, yielding profits all along. The second term is the forward-looking expected value of efforts undertaken in the future. Crucially, it is not affected by the fundamentals: both the marginal effect of effort on fundamentals, $(r+\alpha)\,dt$, and the marginal value of fundamentals, $\frac{1}{2(r+\alpha)}$, are constant. Thus, Markov strategy does not need to condition on the past and has partners exert constant effort, a_M, which equates marginal cost with the constant Markov incentives of one half.

Our assumptions on the cost of effort normalize both the level of effort, as well as the continuation payoffs in the Markov equilibrium, to zero. We say that a partnership *unravels* if, from that point on, partners exert no more effort—that is, partners play the Markov equilibrium.

We highlight that the Markov equilibrium is inefficient. As a partner's effort benefits the two partners equally, the marginal social benefit of effort is one, and so twice higher than the incentives in the Markov equilibrium. In the rest of the paper, we show how non-Markovian, relational incentives may bridge part of this efficiency gap.

Strongly Symmetric Equilibria The only information about the partners' efforts is provided by the stream of profits. As both partners' efforts enter profits additively, it is not possible to identify which of the partners did, and which one did not, contribute to the common good (Fudenberg, Levine, and Maskin (1994)). Thus, as in the classic analysis of repeated duopoly by Green and Porter (1984) or of partnerships by Radner, Myerson, and Maskin (1986), it is not possible to provide incentives by "transferring" continuation value between the agents via asymmetric play, shifting resources from likely deviators.

Therefore, we concentrate throughout the paper on equilibria in which players choose symmetric strategies, conditioning the provision of effort on the public history available to them in the same way. Formally, a *strongly symmetric equilibrium (SSE)* is a PPE in which the strategies $\{a_t^1, a_t^2\}$ satisfy $a_\tau^1 \equiv a_\tau^2$, after every public history in $\mathcal{F}_t$.

Accounting of Incentives and Local Strongly Symmetric Equilibria Define *relational capital* w_τ as the expected discounted payoff from future (symmetric) efforts, that is, the continuation value net of the expected value of fundamentals,

$$w_\tau := W_\tau - \frac{1}{2\,(r+\alpha)}\mathbb{E}_\tau^{\{a_t,a_t\}}[\mu_\tau] = \mathbb{E}_\tau^{\{a_t,a_t\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\,(a_t - c(a_t))\,dt\right]. \tag{7}$$

Relational incentives are constructed by conditioning future play on public signals. They are provided by players "burning value": coordinating on future effort profiles leading to relatively high relational capital after "good" signals—indicating high past effort—and on relatively low relational capital after "bad" signals. When a partner increases effort for a short interval of time, the probability distribution of future histories changes, shifting towards the ones with high relational capital.

Specifically, we define *relational incentive* F_τ as the marginal benefit of effort net of Markov incentives, or, equivalently, as the marginal effect of effort on relational capital. Formally,

$$F_\tau := \frac{\partial}{\partial\varepsilon}\mathbb{E}_\tau^{\{a_t,a_t\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\,(a_t - c(a_t))\,dt\right], \tag{8}$$

for the revenue processes $dY_t^\varepsilon = \overline{\mu}_t^\varepsilon dt + \sigma_Y dB_t$, where $\overline{\mu}_\tau^\varepsilon = \overline{\mu}_\tau + \varepsilon\,(r+\alpha)$ is the expected fundamentals initially increased by an infinitesimal extra effort (see formula (1)), for $\varepsilon > 0$, which subsequently evolve analogously to $\overline{\mu}_t$,

$$d\overline{\mu}_t^\varepsilon = (r+\alpha)\,(a_t^1 + a_t^2)dt - \alpha\overline{\mu}_t^\varepsilon dt + \gamma_t[dY_t - \overline{\mu}_t^\varepsilon dt],\ \ t \geq \tau. \tag{9}$$

A necessary condition for a symmetric equilibrium is that effort is locally optimal, and the partners cannot profit by a small change in effort at any moment on the equilibrium path. That is, given Markov incentives of one half and relational incentives F_τ,

$$a(F_\tau) = \arg\max_a \{(F_\tau + 1/2) \times a - c(a)\}. \tag{10}$$

A *local strongly symmetric equilibrium (local SSE)* is a profile of symmetric strategies such that, following any history, actions are locally optimal, $a_\tau = a(F_\tau)$, for the function $a(\cdot)$ defined in (10), and F_τ as in (8). Finally, let $\mathcal{E}$ be the set of relational capital-incentive pairs (w_t, F_t) achievable in a local SSE. We parametrize its boundary of supremum incentives by F.[15]

[15]F, with $F(w) = \{sup\ F_t|(w_t, F_t) \in \mathcal{E}\}$ is a partial function, defined only over the relational capitals

3 Solution

This section contains the main technical results of the paper. It characterizes the set of relational incentives and relational capitals that can be delivered in a local SSE. It also constructs (nearly) optimal local SSE, and provides conditions for them to be fully incentive compatible.

As the first step, the following lemma shows how relational capital and relational incentives must evolve in a local SSE. Throughout this section, fundamentals are imperfectly monitored, $\sigma_Y > 0$.

Lemma 1 *A symmetric strategy profile $\{a_t, a_t\}$ with bounded relational capital and relational incentives processes $\{w_t\}$ and $\{F_t\}$ is a local SSE if and only if there is a L^2 process $\{I_t\}$ such that*

$$\begin{aligned} dw_t &= (rw_t - (a_t - c(a_t)))\, dt + I_t \times (dY_t - \overline{\mu}_t dt) + dM_t^w, \\ F_\tau &= \mathbb{E}_\tau^{\{a_t,a_t\}} \left[\int_\tau^\infty e^{-(r+\alpha+\gamma)(t-\tau)} \, (r+\alpha)\, I_t dt \right], \end{aligned} \tag{11}$$

and actions satisfy $a_t = a(F_t)$, where $\{M_t^w\}$ is a martingale orthogonal to $\{Y_t\}$.

The first equation in the lemma is a version of the "promise keeping" accounting for the continuation value (see Sannikov (2007)). If the current flow of relational capital is lower than the average promised flow, then the relational capital must deterministically increase in the next period, and vice versa. Moreover, relational capital also changes stochastically in response to the unexpected profit realizations, $dY_t - \overline{\mu}_t dt$, with linear sensitivity I_t. The martingale process captures the possibility of public randomization.

A key intuition behind the second equation in (11) is that a deviation to a higher effort today results in fundamentals above the publicly expected level not only now, but throughout the future (see, also, Prat and Jovanovic (2014), Sannikov (2014), and Prat (2015)). This means unexpectedly good news—profits higher than expected—throughout the future, which keep pushing relational capital up (when sensitivities I_t are positive).

achievable in a local SSE.

After a deviating effort, the wedge between the private and public expectation of the fundamentals reverts to zero gradually, at a rate $\alpha + \gamma$. The first term is the exogenous rate of decay of the fundamentals and the second term is the endogenous speed of learning from profits about the fundamentals (Equation (3)). As an example, consider an off-equilibrium increase in effort. Naturally, this leads to a stream of unexpectedly high profits. Upon observing it, the public attributes part of the higher profits to a persistent change in the partnership's quality (due to it being stochastic) and part of it to transient luck during this period (due to imperfect monitoring). The first effect is incorporated into higher expectations of the fundamentals, and so higher expectations of profits. Hence, as the stream of higher-than-expected profits is realized, the wedge between private and public expectations shrinks. The effect of learning is, thus, that, following the off-equilibrium higher effort, the realized higher-than-expected profits are gradually attributed to a persistent exogenous change in the partnership's quality (in a similar fashion to Holmström (1999) and Cisternas (2017)). Note that when fundamentals are deterministic, $\sigma_\mu = 0$, partners do not learn in equilibrium, $\gamma = 0$. We discuss the effect of learning on relational incentives in more detail in Section 4.2.

Lemma 1 identifies relational capital and incentives as two variables that characterize any local SSE. The following result establishes an HJB characterization of F, hence obtaining the boundary of the set of relational capital-incentive pairs achievable in a local SSE.

Theorem 1 *The upper boundary F of relational incentives achievable in a local SSE is concave and satisfies the differential equation*

$$\begin{aligned}(r+\alpha+\gamma)F(w) &= \max_I \left\{(r+\alpha)I + F'(w)\left(rw - [a(F(w)) - c(a(F(w)))]\right) + \frac{F''(w)\sigma_Y^2}{2}I^2\right\} \\ &= F'(w)\left(rw - [a(F(w)) - c(a(F(w)))]\right) - \frac{(r+\alpha)^2}{2\sigma_Y^2 F''(w)}, \qquad (12)\end{aligned}$$

on $[0, w^*)$, *as well as the boundary conditions*

$$\begin{aligned} F(0) &= 0, \\ \lim_{w \uparrow w^*} \left\{ (r+\alpha+\gamma)F(w) - F'(w)\left(rw - [a(F(w)) - c(a(F(w)))]\right) \right\} &= 0, \\ \lim_{w \uparrow w^*} \left\{ rw - [a(F(w)) - c(a(F(w)))] \right\} &= 0, \end{aligned} \tag{13}$$

where w^* *is the supremum of the relational capitals achievable in a local SSE. Moreover,* w^* *is not attained by any local SSE.*

Theorem 1 provides a characterization in the form of a differential equation and boundary conditions of the supremum F of relational incentives achievable in a local SSE. In the first line of equation (12), the left-hand side is the average flow of relational incentives needed to generate the stock of relational incentives $F(w)$, given the exponential discounting, mean reversion, and learning. The right-hand-side of (12) has the form of an HJB equation for the problem of maximizing relational incentives F as a function of relational capital. It is the point-wise maximum, over all policies I, of the expected flow of relational incentives plus the stochastic differential operator applied to F. Specifically, the first term on the right-hand side captures the flow of relational incentives. This reflects how responsive the relational capital is to the public signals. The second term captures the change in the relational incentives resulting from the drift in the relational capital. Relational incentives increase if the relational capital drifts in the direction where the optimal incentives are higher. Finally, the last term captures the loss resulting from the second-order variation in the relational capital. The (instantaneous) variance σ_Y^2 of the signal translates to (instantaneous) variance $\sigma_Y^2 I^2$ of the relational capital, which decreases future incentives due to the concavity of F.

The first boundary condition in (13) says that the relational incentives in any local SSE with no relational capital must be zero. Strictly positive incentives, with equilibrium effort by a partner, would allow for a profitable deviation to shirking. The second boundary condition is equivalent to the second derivative of F being unbounded and, thus, the optimal sensitivity, I, dying out as one approaches the bliss point. As a consequence, close to the bliss point, good outcomes are not rewarded and bad outcomes are not punished: the flow of relational incentives dies out. The last boundary condition requires the drift of the relational capital to also die out. The arguments behind the last two boundary conditions are related. Either a positive drift or volatility would lead to an escape of the relational capital beyond the supremum. Moreover, if the drift were strictly negative, one could generate relational capital above the supremum, simply by letting it drift down.

We note that the representation of the relational capital and relational incentives in Lemma 1 have a form of a standard stochastic control problem. The Lemma provides a law of motion of the state variable (relational capital) and an objective function in the form of an integral (relational incentives), which depends on a policy function (sensitivity I_t). However, in the representation in Lemma 1, the law of motion of the state variable depends on the level of the objective function, via effort chosen, as in equation (10). While such dependence is not allowed in a formulation of a standard stochastic control problem, the proof in the Appendix establishes that the HJB characterization of the solution via the HJB differential equation as in the theorem is valid.

Theorem 1 also shows that the supremum relational capital is not attainable, and so an optimal local SSE does not exist. This follows from the last two boundary conditions in (13), which imply that the supremum relational capital would have to be the outcome of a stationary—and, hence, Markov—equilibrium.[16] The construction of near-optimal local SSE is achieved in the following result.

To guarantee existence, we now restrict attention to a class of local SSE, with sensitivities I_t of relational capital with respect to profit flow either zero or weakly above ε, for $\varepsilon > 0$. This yields self-generation of the upper boundary F_ε in the following theorem.[17] A local SSE is called *ε-optimal* if it belongs to such class and gives rise to relational capital

[16]Formally, the non-existence of the optimal local SSE hinges on the openness of the set of strictly positive yet arbitrarily small policies $\{I_t\}$, in the local SSE that approximate the unattainable supremum w^* (second boundary condition in 13).

[17]We point out that there might be other types of approximately optimal local SSE. The particular class we focus on has the additional benefit of numerical tractability; in Appendix A.3 we provide the bounds on the first two derivatives of the functions F_ε in Theorem 2. In contrast, note that the equation (12) in Theorem 1 has F'' arbitrarily small close to the right boundary.

close to the supremum.[18]

Theorem 2 *For $\varepsilon > 0$, consider local SSE with sensitivities I_t of relational capital with respect to profit flow either zero or weakly above ε. The upper boundary F_ε of relational incentives achievable in a local SSE under this constraint is concave and satisfies the differential equation*

$$(r+\alpha+\gamma)F_\varepsilon(w) = \max_{I_\varepsilon \in [\varepsilon,\infty)} \left\{ (r+\alpha)I_\varepsilon + F'_\varepsilon(w)\left(rw - [a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))]\right) + \frac{F''_\varepsilon(w)\sigma_Y^2}{2} I_\varepsilon^2 \right\} \quad (14)$$

on $[0, w^*_\varepsilon]$, as well as the boundary conditions

$$\begin{aligned} F(0) &= 0, \\ (r+\alpha+\gamma)F(w^*_\varepsilon) - F'(w^*_\varepsilon)\left(rw^*_\varepsilon - [a(F(w^*_\varepsilon)) - c(a(F(w^*_\varepsilon)))]\right) &= 0, \\ rw^*_\varepsilon - [a(F(w^*_\varepsilon)) - c(a(F(w^*_\varepsilon)))] &< 0, \end{aligned} \quad (15)$$

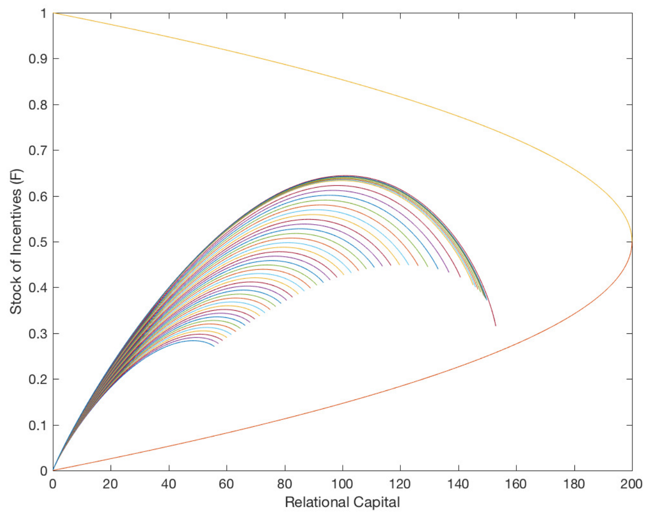
where w^*_ε is relational capital achieved in an ε-optimal local SSE. Moreover, for any solution of differential equation (14) with boundary conditions (15) on an interval $[0, w_\varepsilon]$, there is a local SSE achieving w_ε.

The result provides a tool to find the (approximate) supremum of relational capitals achievable in local SSE. Any function solving the differential equation together with the boundary conditions defines an achievable level of relational capital. The right-most argument of the solution that reaches furthest to the right is the (approximate) supremum.

The function F_ε in the theorem provides a recipe for constructing near-optimal local SSE (see Lemma 2 in the next section). In Figure 1, F_ε is the highest inverse parabola, which reaches furthest to the right. At any point in time, for any value of relational capital, the function determines relational incentives, and so the marginal benefit of effort. This pins down the equilibrium effort and also the relational capital in the next instant: it drifts deterministically—say, decreases if the flow benefits are large relative

[18] Formally, we require the distance to the supremum to be vanishing in ε. The equilibria in the next theorem, in particular, achieves distance of order $O(\varepsilon^{1/3})$.

to the relational capital—but also responds to the stochastic news about the profit flows (see Lemma (1)). The sensitivity to those news is the one that maximizes expression (14) and, again, is pinned down by function F_ε and its second derivative. In the next instant, the game continues with updated relational capital (as described) and beliefs about the fundamentals (see Equation 3).



This figure displays many different solutions of the differential equation (14), with the near-optimal local SSE characterized by the curve that reaches farthest to the right. The horizontal parabola is the locus of the feasible relational capital-incentives pairs (w, F) that can be achieved by symmetric play in a stage game, satisfying $rw = a(F) - c(a(F))$. The efficient pair is $(200, 1/2)$.

Figure 1: Relational Incentives in a Near-optimal SSE

3.1 Global Incentive Compatibility

So far, we have characterized local equilibria. The following result shows conditions on the primitives, under which local SSE satisfy full incentive-compatibility constraints. For simplicity, in the remaining results we assume that the cost of effort is quadratic:[19]

$$\text{(Quadratic Cost)} \qquad c(a) = \frac{1}{2}a + \frac{C}{2}a^2. \tag{16}$$

Theorem 3 *Fix $\varepsilon > 0$ and consider an $\varepsilon-$optimal local SSE $\{a_t, a_t\}$. Then, $\{a_t, a_t\}$ is an SSE when $C\sigma_Y$ is sufficiently high, where C is the second derivative of the cost function and σ_Y is observational noise.*

The problem in establishing global incentive compatibility consists of showing that, after any history, the effort choice is concave. Given that the effort cost function is strictly convex, with the second derivative C, this boils down to establishing bounds on how convex the expected benefit of effort can be. Crucially, in a dynamic environment with persistence, like ours, a deviation affects the strength of incentives that the agent faces in the future. This knock-on effect makes accounting for the benefits of deviations much more involved than in a static setting, or without persistence.

Aligned with this intuition, to bound how convex the benefit of effort is, we establish a uniform bound on how sensitive the relational incentives are with respect to public signals for any history. The proof of Theorem 3 can thus be divided into two parts. The first—related to results in the literature—shows that there are no global deviations from a local SSE if the sensitivity of relational incentives is uniformly bounded (see Williams (2011), Edmans et al. (2012), Sannikov (2014), and Cisternas (2017)). The second part of the proof provides a bound for this endogenous sensitivity of relational incentives, as a function of the primitives of the model. This relies on the analytical tractability of our solution, for a given $\varepsilon > 0$.

[19] Quadratic costs greatly simplify deriving the bounds in Theorem 3 and the propositions in Section 4, but we are confident that the result can be extended to more general cost functions. Equation (39) in Appendix A.4 provides a precise sufficient condition on the parameters that guarantees global incentive compatibility.

4 Information Structure and Partnership

The informational environment of a partnership, in our setting, is determined by three parameters (see Section 2.1). First, partnerships differ by how persistent are the effects of partners' efforts and decisions. This is captured by the speed of depreciation, or mean-reversion of the fundamentals, α, with high mean-reversion corresponding to transient efforts, as in a repeated game. Second, partnerships differ by how well their fundamentals are monitored, and so how well the progress of the venture can be tracked and assessed. This is captured by the degree of noise in the public signals, σ_Y. Third, partnerships differ by the level of uncertainty about the quality of the venture or the partners. This is captured by the degree of volatility of the fundamentals, σ_μ.

In this section, we investigate separately how the three dimensions of the information structure affect the partnership's value. (Note that efforts in the Markov equilibrium do not depend on the information structure; see Proposition 1). We begin by establishing the conditions on the informational environment for the existence of non-trivial equilibrium. At the end we extend the model to allow for career concerns and compare the effects of the informational environment on career and relational incentives.

4.1 Persistence and Non-Trivial Equilibria

Relational incentives are discounted at a rate of $r + \alpha + \gamma$, which accounts for the time preference, the persistence of the fundamentals, and the speed of learning. The next proposition establishes that this "discount rate" is the key parameter determining existence of SSE with nontrivial relational incentives, which improve upon the Markov Equilibria.[20]

Let w_{EF} be the efficient level of relational capital, i.e., the relational capital resulting from the constant efficient effort a_{EF} by each partner—i.e. $c'(a_{EF}) = 1$.

[20] The exact parameter bounds for the existence result—part (i) of the proposition below—can be found in Equations (41) and (43) in Appendix A.5. Formula (50) in Appendix A.5 describes the speed of convergence for part (ii).

Proposition 2 *i) Fix the ratio $\frac{r+\alpha+\gamma}{r}$. There is $\delta > 0$ such that when $r + \alpha + \gamma$ is sufficiently small then the supremum of relation capitals achievable in an SSE is at least $\delta \times w_{EF}$.*

ii) In contrast, for every $\delta > 0$ the supremum of relational capitals achievable in an SSE is less than $\delta \times w_{EF}$, when $r + \alpha + \gamma$ is sufficiently large.

The "discount rate" $r+\alpha+\gamma$ determines the time horizon for the provision of incentives (see Proposition 1). To motivate today's effort when the rate is high, high-profit outcomes must be rewarded soon—either because partners do not care much about the future (r high), the effect of effort on profits wears off quickly, as when approximating a repeated game (α high), or the effect is quickly attributed to a change in partnership's quality (γ high).

Relational incentives die out when the "discount rate" is high enough for two reasons. The first reason is specific to relational incentives: when the relationship is at the bliss point of highest relational capital, no more relational rewards are available for the partners. The second reason is that over a short interval of time, profit signals about the efforts are weak. Even with such imprecise signals, incentives would be possible by combining rewards and punishments: while rewards and punishments are often meted out incorrectly, the two errors may cancel out. However, incentives cannot be provided solely with punishments, as they would be used mistakenly in equilibrium too often (see Abreu, Milgrom, and Pearce (1991) and Sannikov and Skrzypacz (2007, 2010)). Taken together, when the "discount factor" is high and so the time horizon for incentive provision is short, relational incentives cannot be sustained at the bliss point of the partnership. The construction of relational incentives essentially unravels.

In contrast, with a low "discount rate" on incentives, nontrivial relational incentives are possible. Good signals (i.e high profit outcomes) are rewarded when the relational capital of the partnership is in interior ranges. If the relationship is at the bliss point, partners exert effort because such effort will be rewarded later, once the relational capital drifts down. Waiting does not destroy much of the incentives since the discounting is low.

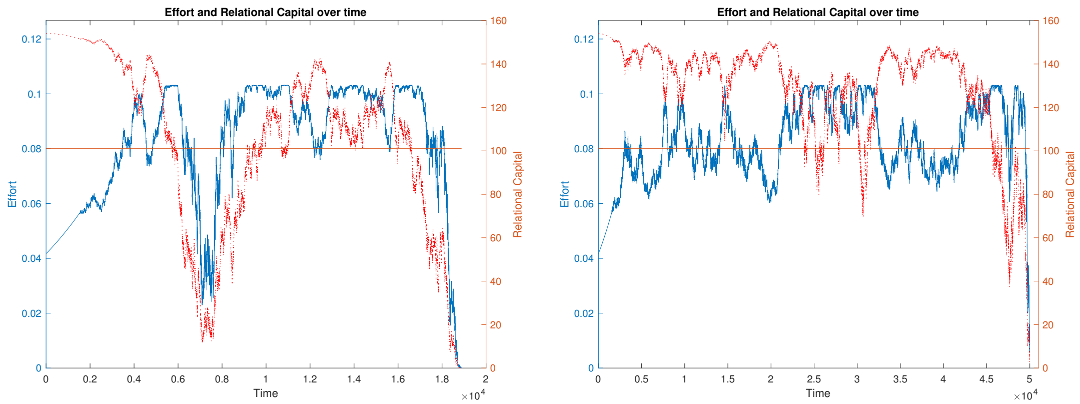
Relational incentives may, thus, increase when the effects of partners' efforts are more long-lasting and persistent, with low α. One interpretation of this result is that partnerships may be the appropriate form of organizing a firm when partners make strategic decisions that determine the future of the venture (α low), rather than routine decisions that execute well-established blueprints to keep the revenue flowing (α high), as in a repeated game.

Dynamics. The impossibility in the previous proposition is driven by the impossibility of a flow of incentives near the bliss point of the partnership. This structure of incentives also affects the equilibrium dynamics of effort in a non-trivial near-optimal equilibrium. Profit outcomes that exceed expectations are always good news for the partnership, increasing relational capital, as $I_t \geq 0$. However, they do not always lead to greater effort.

On one extreme, when a partnership runs out of relational capital, the partnership unravels, relational incentives disappear, and no effort is taken in the future. Near this extreme, when relational capital is low, a good outcome that increases relational capital prolongs the life of a partnership and hence increases relational incentives and effort. Formally, function F_ϵ is increasing in this range.

On the other extreme, close to the bliss point, the flow of incentives vanishes and partners are motivated by future increases in the relational incentives, once relational incentives drift down. A good outcome only postpones the arrival at the interior ranges, and hence decreases relational incentives and effort. Formally, function F_ϵ is decreasing in this range.

Corollary 1 *In a near-optimal local SSE there is a threshold level of relational capital, $w^{\#}$, such that i) at relational capitals below $w^{\#}$ high profit realizations dY_t increase equilibrium effort ("rallying"), and ii) at relational capitals above $w^{\#}$ high profit realizations dY_t decrease equilibrium effort ("coasting").*



(a) A short-lived partnership (b) A long-lived partnership

Each panel displays a sample path of effort (on the left axis) and relational capital (on the right axis) over time, starting near the supremum relational capital. The horizontal line represents the level of relational capital at which effort is maximized. Initially players coast, and the relational capital drifts down, undisturbed by shocks. When relational capital is above the horizontal line, profit outcomes that increase relational capital lead players to exert less effort. Changes in effort and relational capital are negatively correlated. When relational capital is below the horizontal line, changes in effort and relational capital are positively correlated.

Figure 2: Effort and Relational Capital over Time

4.2 Monitoring the Partnership

The equilibrium in our dynamic environment is inefficient because partners do not observe each other's effort. Otherwise, they could incentivize efficient effort by reverting to the inefficient Markov equilibrium following any spat of shirking.[21] It seems thus compelling that better monitoring of the fundamentals in our game should increase the partnership's value. The following proposition shows that this intuition captures only part of the story.[22]

Proposition 3 *i) Suppose fundamentals are deterministic, $\sigma_\mu = 0$. The supremum w^* of relational capital achievable in a local SSE is increasing in the precision of the monitoring technology σ_Y^{-1}.*

[21]Modeling a continuous-time game with perfect monitoring runs into the usual problems. However, "Grim-Trigger" strategies approximate efficiency in a discrete-time approximation of the game, given that periods are "short" and so discount factor arbitrarily close to one, and the MinMax strategy is a stage-game Nash equilibrium.

[22]For a given level of fundamental noise σ_μ, formula (51) in Appendix A.5 provides the speed of convergence for part ii) of the proposition.

ii) Suppose the fundamentals are stochastic, $\sigma_\mu > 0$. The supremum w^* of relational capital achievable in a local SSE converges to zero, when the precision of the monitoring technology σ_Y^{-1} converges to zero.

iii) Suppose the fundamentals are stochastic, $\sigma_\mu > 0$, and monitored perfectly, $\sigma_Y = 0$. The unique SSE is the Markov equilibrium, with relational capital $w = 0$.

When there is no uncertainty about the quality of the partnership and fundamentals are deterministic, better monitoring always improves efficiency (part (i) of the proposition). The intuition is simple: absent uncertainty about the quality, the public signals are used solely as signals of effort, and better monitoring mitigates informational frictions.[23]

When the quality of the fundamentals is uncertain, partners use public signals not only to incentivize effort but also to estimate the quality of the partnership. Better monitoring still benefits the partnership by providing better signals of efforts, but it also results in better learning about the fundamentals (higher gain parameter γ). Crucially, faster learning means that good outcomes are quickly incorporated in increased expected fundamentals, and the window for rewarding unexpectedly high outcomes, and so effort, shrinks. This shorter horizon for the incentive provision is particularly harmful at the bliss point of the partnership, when effort can be motivated solely by future rewards (Proposition 2). Part (ii) of Proposition 3 establishes that, with little noise, this negative effect is dominant and eliminates relational incentives.[24]

In the extreme, if fundamentals are monitored with no noise, the current change in fundamentals is a sufficient statistic to evaluate current effort. Incentives must be provided immediately, as in the repeated game i.i.d. setting. Since this is not possible at

[23]Note that providing the same level of incentives with less noise requires less variability of relational capital in equilibrium. Suppose $\sigma_\mu, r = 1,\ \alpha = 0$; generating relational incentives F of, say, one, requires sensitivity I of relational capital to public signal equal one as well. This results in the volatility of relational capital σ_Y, increasing in noise. Formally, the only impact of σ_Y on the HJB equation (12) is through the last term, with the cost of incentives due to the second-order variation of relational capital increasing in σ_Y.

[24]Note that, as σ_Y shrinks, only the left-hand side (required mean flow of incentives) and the last term in the HJB equation 12 (contribution of the incentive flow) are scaled up. When the middle term capturing the benefit of delayed incentives disappears, the effect is similar as when the horizon for incentive provision shrinks. Formally, the solution of the HJB equation that starts around $w^* > 0$ would reach arbitrarily high levels, since i) $F''(w)$ is bounded away from negative infinity, as long as $F(w)$ is bounded away from zero; and ii) $F'(w)$ is arbitrarily steep close to w^* (Theorem 1).

the bliss point of maximal relational capital, the construction of any relational incentives unravels (part (iii) of the proposition). This result is related to the impossibility of nontrivial incentives in Sannikov and Skrzypacz (2007, 2010).[25]

One solution to this impossibility, following Abreu, Milgrom, and Pearce (1991), is to withhold the arrival of information; players observe the relevant path of signals only at times $l, 2l, 3l$, etc., for a fixed time length $l > 0$. In the new game—with "compounded" periods, actions, and signals—the horizon for incentive provision is still only the current period. However, the bundling of information improves the information quality in any given period and partners can be incentivized to work even when the relationship is at its best.

Proposition 3 above provides an alternative way of delaying the arrival of information: *poorer* monitoring of the state. When effort has persistent effects, increasing the noisiness of the signal of past efforts means that it takes longer for a partner to learn about effort at any particular date. Our results show that with perfectly monitored fundamentals it is not the "short periods", but the instantaneous arrival of the relevant information and instantaneous time horizon for incentives that hampers the provision of incentives in the partnership. The impossibility in Sannikov and Skrzypacz (2007, 2010) is not due to the peculiarities of continuous-time modeling, but to the extreme assumption of perfect state monitoring.

4.3 Uncertainty about the Partnership

In our environment, there is an additional obstacle to the partners' monitoring of each other's efforts: the quality of the partnership is stochastic and unobserved by the partners.

[25]While part iii) of the proposition establishes only the impossibility of nontrivial strongly symmetric equilibria (when fundamentals are observable), the impossibility can be extended to asymmetric PPE, under appropriate conditions. Namely, suppose that the cost of effort satisfies $c''' \leq 0$. The condition implies that for a given sum of efforts, the incentivizing sensitivity of the sum of private relational capitals $w_t^1 + w_t^2$ with respect to profits is minimized with symmetric efforts. The proof of the impossibility of incentives at a bliss point of the partnership is almost analogous to the one in the symmetric case; see Sannikov and Skrzypacz (2007), Section V.B for details.

The uncertainty is captured by the degree of volatility of the fundamentals, σ_μ.[26]

Proposition 4 *The supremum of relational capital, w^*, that the partnership can generate in a local SSE decreases when the uncertainty regarding the partnership quality, σ_μ, increases.*

In contrast to improved monitoring, reducing uncertainty about the quality of the joint venture facilitates the provision of incentives. This is because reduced uncertainty results in the public news more closely tracking the effort exerted by the partners, instead of reflecting the exogenous changes in quality.

One implication of the result is that the use of relational incentives to motivate the partners is more adequate in mature, long-standing relationships, whose quality—technology, product, environment, synergies, etc—is better known.[27] As the partnership is better understood, the partners can use the public news to more precisely reward the provision of effort. In contrast, in young enterprises, the uncertainty about the quality of the joint venture gets confounded with the uncertainty about the level of fundamentals. Hence, if one of the partners free rides, part of the bad news will be attributed to a "worse than expected" quality, inhibiting punishments.

Finally, depending on the level of uncertainty about the venture, a stream of bad outcomes has a differential effect on relational capital and expected fundamentals. In mature partnerships with little uncertainty (σ_μ is low or absent), profit outcomes barely affect the expected fundamentals (as γ is close to zero). On the other hand, relational capital is sensitive to the public news. It follows that a profitable and mature partnership may unravel when its goodwill is tested by a short string of sharp, adverse outcomes, with hardly any effect on its profitability (see Equations (3) and (4)). The Beatles (10 years together) and Daft Punk (28 years) in music; Jamie Dimon and Sandy Weill from Citigroup (15 years) in finance; and Daniel Humm and Will Guidara from Eleven Madison Park (13 years) in fine-dining provide stylized evidence. Younger enterprises, with more

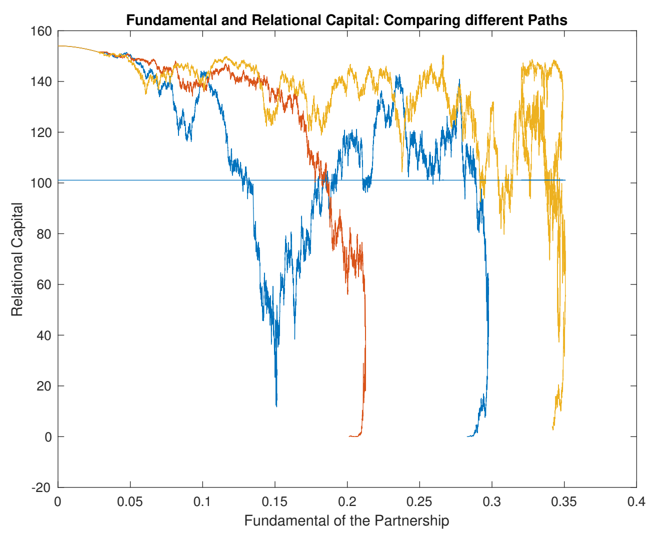
[26] More precisely, variance in beliefs is strictly increasing in σ_μ, end equals zero when σ_μ does; see Equation 4.

[27] For tractability, in this paper, we consider only stationary models, with uncertainty constant over time. We can interpret it to be low for mature ventures, and high for younger ones.

uncertainty about the partnership quality, tend to burn their perceived productivity before dissolution.

Corollary 2 *("Beatles' break-up") In a near-optimal local SSE, at any point in time t, a partnership may unravel in an arbitrarily short period of time after a sequence of unexpected bad news. The accompanying change in the expected profitability is of order σ_μ times the amount of bad news.*

Figure 3 displays the differences in the dynamics of the fundamentals and of the relational capital. It shows three different sample paths, highlighting that a partnership's relational capital is not determined by its profitability. Furthermore, even at dissolution, partnerships have different levels of the fundamentals.



This figure displays three different sample paths of the relational capital of a partnership, as a function of the fundamentals of the relationship. The horizontal line marks the level of relational capital at which effort is maximized.

Figure 3: Relational Capital and Fundamentals of a Partnership

4.4 Relational Incentives versus Reputation

In this paper, we have motivated the partners through relational incentives. However, going beyond our model, the alternative incentive mechanism in weak-contractual environments is to motivate agents by reputational effects, i.e., working to build a good name in the market (*career concerns*, Holmström (1999)).[28] Our model can be readily extended to allow for career concerns. Importantly, the comparative statics of the effects of the informational environment on career concerns is very different from the effects on relational incentives, established above.

When building a reputation, a partner exerts effort so that good profit outcomes are misattributed to a high quality of the venture, which in turn increases the venture's market value, captured by the partners. We may accommodate career concerns in our model by adjusting the cumulative profit flow of the partnership at time t, π_t, to consist of the weighted sum of the noisy signal (as before) and the public belief of the partnership fundamentals, $\pi_t = (1-\kappa)dY_t + \kappa\overline{\mu}_t$, with the parameter $\kappa \in [0,1]$ capturing the relative weight given to the public belief. In our baseline model, $\kappa = 0$.[29]

Compared to the direct incentives in the main model, career concerns incentives are scaled by $\frac{\gamma}{r+\alpha+\gamma}$. This is because an extra unit of effort increases the private expectation of the fundamentals above the public expectation, as in the main model. The difference in the expectations degrades over time at a rate $\alpha+\gamma$, and raises the payoff relevant parameter (public expectation) by a factor of γ, at any point in time. Finally, an increase in the payoff relevant parameter degrades over time as in the main model.[30]

It follows that, for a fixed set of parameters, the analysis of the provision of incentives in a partnership with career concerns proceeds analogously as in the main model, but with Markov incentives adjusted to $\frac{\kappa\gamma}{2(r+\alpha+\gamma)} + \frac{(1-\kappa)}{2}$. Unlike in the main model, however, Markov incentives are now affected by the information structure parameters $\alpha, \sigma_Y, \sigma_\mu$ (in line with Holmström (1999)).

[28]Note that unlike in Holmström (1999) signal-jamming model, in our model efforts affect the state rather than the signal.

[29]In Holmstrom's model, the venture can be sold in every period in a competitive market, giving the owner rights to collect the profit flow. The competitive price is the publicly expected level of the profit flows, i.e., the fundamentals, $\overline{\mu}_t$. Thus, κ may be interpreted as the fraction of the partnership traded.

[30]Combining the three effects, the total marginal benefit of effort F_{cc} due to career concerns is

$$F_{cc} = (r+\alpha) \times \frac{\gamma}{r+\alpha+\gamma} \times \frac{\kappa}{2(r+\alpha)} = \frac{\kappa\gamma}{2(r+\alpha+\gamma)}.$$

Corollary 3 *With career concerns, Markov incentives are i) increasing in persistence α^{-1}, ii) increasing in the precision of the monitoring technology σ_Y^{-1}, and iii) increasing in the uncertainty about the partnership quality σ_μ.*

While persistence has a similar effect on career concerns and relational incentives, the effects of the monitoring technology and of uncertainty about the quality are in direct contrast in the two settings. First, while precise monitoring eliminates relational incentives, it improves learning about the changes in the quality of the partnership, precipitating the arrival of the career concerns rewards. Second, the mechanism through which the uncertainty affects the two kinds of incentives is the same, yet with opposing effects. High uncertainty crowds out the relational reward for good outcomes, which are misattributed to an exogenous change in the quality. While this restricts relational incentives, the same mechanism facilitates career-concern incentives for effort.

These contrasting comparative statics have implications on the cross-section of organizational structures. Partnerships benefit when the progress of the venture is hard to monitor, i.e. based on long-term, qualitative contributions, but the uncertainty about the venture is low. In contrast, reliance on reputation-based incentives is more fruitful if progress is easier to quantify and measure, but the uncertainty regarding the venture quality, and so the scope for building reputation, is high. This is consistent with the observation that partnerships are very common in the professional sector, i.e. law firms, accounting, and advertising (see Levin and Tadelis (2005) and Von Nordenflycht (2010)). The sector is dominated by old, established firms, with little outstanding uncertainty about them, and at the same time the product of a firm is opaque.[31]

[31] In particular, in these knowledge-intensive environments, even after the output is produced and delivered, its quality is hard to evaluate (see Empson (2001), Greenwood and Empson (2003), Broschak (2004), and Von Nordenflycht (2010)). For instance, for an advertising agency, even after the campaign is published its quality and effects are hard to measure: Was the advertising agency's campaign responsible for the sales increase? A similar argument holds for other professional partnerships, i.e. was the lawyer's argument responsible for the acquittal?

5 Concluding Remarks and Discussion

The paper characterizes optimal relational incentives in a model of partnerships, with persistent effects of partners' efforts and imperfect state monitoring. The main economic result is a novel mechanism for the benefit of poorer monitoring, specific to relational incentives, which allows rewards and punishments to be delivered in a longer window of time. Overall, our insights are consistent with the predominance of well-established partnerships in industries in which the effects of effort are long-lived and hard to measure (such as in the professional sector).

The insight also opens up the analysis of relational incentives in a continuous-time setting. Persistence and imperfect state monitoring are important for applications, yet are known to lead to intractable solutions. We develop a method to characterize optimal relational incentives that extends stochastic control to game-theoretical settings with persistence and imperfect state monitoring.

There are many ways in which one may modify our benchmark model. We conclude the paper by informally discussing the robustness of the results with respect to some of them.

1) Frequency of Play. One question is whether the results of the paper are specific to the continuous-time Brownian setting considered here, or if they hold in an approximate model with short, discrete time periods and Normal noise.

We have no reason to doubt that our results are approximated in the discrete-time setting (albeit at the cost of working with difference equations). First, the boundary of the set of incentives and relational values should be self-generating, and approximately satisfy the differential HJB characterization, with short, discrete time periods. This mirrors the classic results on Folk Theorem (see, e.g., Fudenberg, Levine, and Maskin (1994)), or the results for a Brownian Principal-Agent model (Sadzik and Stacchetti (2015)).[32, 33] Second, the problems with providing incentives close to the boundary point

[32] In the Folk Theorem setting the boundary of the value set is approximated by a linear function; with Brownian model, the approximation requires a quadratic function.

[33] The assumption of Normality in the approximation is likely important, though; see Sadzik and Stacchetti (2015).

of the supremum relational capital persist with a high frequency of play, for information structures with Normal noise that approximate the Brownian model in this paper. This continuity result is the subject of Sannikov and Skrzypacz (2007, 2010), in the special case of no persistence (or in the case of perfectly monitored fundamentals). While we are convinced the continuity holds also in our more general setting, the formal proof is beyond the scope of this paper.

2) Mixed and Asymmetric Strategies; Other Solution Concepts. Throughout the paper we restricted attention to pure strategy strongly symmetric equilibria. Formally, our representation results suggest that mixing may not happen in equilibrium, as partners face locally linear reward schemes in continuous-time, with a fixed marginal benefit of effort. With no mixing, players have no private information, and hence every strategy, even a deviating strategy, is public.[34]

In a discrete-time approximation, mixed strategies may help identify other players' strategies (see Fudenberg, Levine, and Maskin (1994)). Moreover, extending the solution concept to include private communication would allow conditioning of the play on past mixing and open door to random "testing", with subsequent punishment or reward of the opponent (Kandori and Obara (2006)). For example, a partner may at random times decrease her own effort to the minimum, and let the opponent "pay" for low outputs. The same can be achieved with private, mediated messages (Rahman (2014)). This individual monitoring, in the spirit of Alchian and Demsetz (1972) and standard moral hazard, would provide additional incentives in a partnership.

Short of excluding one partner, as discussed above, asymmetric strategies neither improve the monitoring nor increase efficiency (due to convex costs). It can therefore best be thought of as an instrument for additional "burning value". While we believe asymmetric strategies may lead to better equilibria, explicitly allowing some degree of enhancing or burning of the value, discussed next, seems to us a cleaner and more direct modeling choice to capture the same channel.

[34]Note that a public strategy may implicitly condition on own past (pure) deviating actions, through conditioning on the public information at that point in history.

3) Observable Actions. We may allow partners to exert additional observable effort. It may be productive and drive profits up, or it may be unproductive, with the only effect of "burning value". We conjecture that the only difference in the resulting differential equation (12) is the extra term in the drift of the relational capital. When observable effort is parametrized by the effect o it has on the value of the partnership, in the case when $F' < 0$, partners exert the efficient level of observable effort $\overline{o} > 0$, which drives relational capital down. When relational capital is low and $F' > 0$, partners exert the most unproductive effort $\underline{o} < 0$. This "conspicuous toiling" is best viewed as an investment in relational capital, which moves up quickly in response.[35]

4) Capital accumulation impacting productivity. The model explored in this paper is linear—the evolution of fundamentals is linear in the level of fundamentals and in the exerted efforts, and the expected payoffs are linear in the level of fundamentals. While linearity helps with a tractable characterization of relational incentives in partnerships, the framework with persistence and imperfect state monitoring, as well as the solution method, can naturally be extended to a wider class of non-linear models.

Given the familiar difficulties with learning in nonlinear environments, let us focus on the case in which effort directly determines the fundamentals, however, the impact of effort on fundamentals may not be linear (i.e. $d\mu_t = g(\mu_t, a_t^1 + a_t^2)dt$, with g a continuously differentiable function). We can define the continuation payoffs, as well as the marginal benefits of higher fundamentals, analogously to the main model. However, as the impact of effort on fundamentals is not constant, the marginal benefit of a higher fundamental depends not only on the continuation payoff (as before) but also on the level of fundamentals. We can extend our HJB to encompass these scenarios by introducing the level of fundamentals as an additional state variable.[36]

An important application of a nonlinear model is partnerships with capital accumulation. The provision of effort today may affect the productivity of effort in the future and, hence, the incentives for the provision of it. For instance, when developing a new product,

[35] Similar investment in the value of a partnership has been documented in the equilibrium setting by Fujiwara-Greve and Okuno-Fujiwara (2009) and verified in the lab setting by Lee (2018).

[36] A previous version of this paper provided a formal derivation of the differential equation characterizing the boundary of incentives achievable in local SSE in nonlinear settings.

early efforts to hone in a better design have a significant effect on the productivity of later marketing efforts.

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A Appendix: Proofs

A.1 Lemmas 1, 2, and 3.

In this section we prove three crucial lemmas for the proof of Theorem 1. The first is Lemma 1 from the body of the paper. Lemmas 2 and 3 jointly correspond to the *Verification Theorem* from the stochastic control literature.[37] These two lemmas highlight why we may not rely on the existing verification results.

Proof of Lemma 1. The proof can be split in two parts. First, we establish that for an arbitrary pair of symmetric strategies $\{a_t, a_t\}$, relational capital $\{w_t\}$ follows a process (11), for some L^2 process $\{I_t\}$ and a martingale $\{M_t^w\}$ orthogonal to $\{Y_t\}$. The proof follows similar steps as Proposition 1 in Sannikov (2007). We derive the representation for the relational capital process in (11) in the second step.

The process $\left\{Y_t - \int_0^t \overline{\mu}_s ds\right\}$, scaled by σ_Y, is a Brownian Motion, and the process $\widetilde{w}_t = \int_0^t e^{-rs}\left(a_s - c(a_s)\right) ds + e^{-rt} w_t$ is a martingale. Since efforts, and so $\widetilde{w}_t$ are bounded, it follows from Proposition 3.4.15 in Karatzas (1991) that $\widetilde{w}_t$ equals $\int_0^t e^{-rs} I_s \left(dY_s - \overline{\mu}_s ds\right) + M_t^w$, for an appropriate $\{I_t\}$ and a martingale $\{M_t^w\}$. Differentiating and equating both expressions for $\widetilde{w}_t$ yields the representation.

Conversely, for a bounded process $\{v_t\}$ that satisfies (11), define the process $\widetilde{v}_t = \int_0^t e^{-rs}\left(a_s - c(a_s)\right) ds + e^{-rt} v_t$, together with $\widetilde{w}_t$ as above. Both $\{\widetilde{v}_t\}$ and $\{\widetilde{w}_t\}$ are bounded martingales and so, as their values agree at infinity, they agree after every history. It follows that the processes $\{v_t\}$ and $\{w_t\}$ are the same. This establishes the first step.

Let us now evaluate the marginal benefit of effort, and the marginal relational benefit of effort F_τ in particular. Consider the Brownian Motion $\sigma_Y^{-1}\left\{Y_t - \int_0^t \overline{\mu}_s ds\right\}$. It follows from Girsanov's Theorem that the change in the underlying density measure of the output paths induced by the change in expected fundamentals from $\overline{\mu}_\tau$ to $\overline{\mu}_\tau^\varepsilon = \overline{\mu}_\tau + \varepsilon(r+\alpha)$ is

$$\Gamma_t^\varepsilon = e^{-\frac{1}{2}\int_\tau^t \frac{\left(\overline{\mu}_s^\varepsilon - \overline{\mu}_s\right)^2}{\sigma_Y^2} ds + \int_\tau^t \frac{\overline{\mu}_s^\varepsilon - \overline{\mu}_s}{\sigma_Y} \frac{dY_s - \overline{\mu}_s ds}{\sigma_Y}}, \tag{17}$$

for $t > \tau$, where $\{\overline{\mu}_s\}_{s\geq\tau}$ and $\{\overline{\mu}_s^\varepsilon\}_{s\geq\tau}$ are the associated paths of estimates, defined in (3) and (9), with $\overline{\mu}_s^\varepsilon - \overline{\mu}_s = \varepsilon e^{-(\alpha+\gamma)(s-\tau)}$, $s > \tau$. The relational capital at time τ thus

[37] See, e.g., Yong and Zhou (1999), Theorem 5.5.1, for a textbook treatment.

changes to

$$\mathbb{E}_{\tau}^{\{a_t,a_t\}}\left[\int_{\tau}^{\infty} e^{-r(t-\tau)}\Gamma_t^{\varepsilon}\left(a_t - c(a_t)\right)dt\right]. \tag{18}$$

Since

$$\frac{\partial}{\partial\varepsilon}\Gamma_t^{\varepsilon}\bigg|_{\varepsilon=0} = (r+\alpha)\int_{\tau}^{t} e^{-(\alpha+\gamma)(s-\tau)}\frac{dY_s - \overline{\mu}_s ds}{\sigma_Y^2},$$

which is also continuous, it follows that

$$\begin{aligned}
F_\tau &= \frac{\partial}{\partial\varepsilon}\mathbb{E}_{\tau}^{\{a_t,a_t\}}\left[\int_{\tau}^{\infty} e^{-r(t-\tau)}\Gamma_t^{\varepsilon}\left(a_t - c(a_t)\right)dt\right] \\
&= (r+\alpha)\,\mathbb{E}_{\tau}^{\{a_t,a_t\}}\left[\int_{\tau}^{\infty} e^{-r(t-\tau)}\left(a_t - c(a_t)\right)\left(\int_{\tau}^{t} e^{-(\alpha+\gamma)(s-\tau)}\frac{dY_s - \overline{\mu}_s ds}{\sigma_Y^2}\right)dt\right] \\
&= (r+\alpha)\,\mathbb{E}_{\tau}^{\{a_t,a_t\}}\left[\int_{\tau}^{\infty}\left(\int_{t}^{\infty} e^{-r(s-t)}\left(a_s - c(a_s)\right)ds\right)e^{-(r+\alpha+\gamma)(t-\tau)}\frac{dY_t - \overline{\mu}_t dt}{\sigma_Y^2}\right],
\end{aligned}$$

where the last equality follows from the change of integration.

Intuitively, in the last integral above, the inside integral corresponds to the forward looking relational capital, which is then multiplied by a Brownian innovation, scaled by the discounted impact of shifted (expected) fundamentals. The correlation between the relational capital and the Brownian innovation equals I_t, from the representation of the relational capital. This yields F_τ as the expected discounted integral of I_t.

Formally, for fixed τ and $\tau' \geq \tau$,

$$\begin{aligned}
\Phi_{\tau'} &:= (r+\alpha)\,\mathbb{E}_{\tau'}^{\{a_t,a_t\}}\left[\int_{\tau}^{\infty}\left(\int_{t}^{\infty} e^{-r(s-t)}\left(a_s - c(a_s)\right)ds\right)e^{-(r+\alpha+\gamma)(t-\tau)}\frac{dY_t - \overline{\mu}_t dt}{\sigma_Y^2}\right] \\
&= (r+\alpha)\left[\int_{\tau}^{\tau'}\left(\int_{t}^{\tau'} e^{-r(s-t)}\left(a_s - c(a_s)\right)ds\right)e^{-(r+\alpha+\gamma)(t-\tau)}\frac{dY_t - \overline{\mu}_t dt}{\sigma_Y^2}\right] \\
&+ (r+\alpha)\,w_{\tau'} \times \left[\int_{\tau}^{\tau'} e^{-r(\tau'-t)}e^{-(r+\alpha+\gamma)(t-\tau)}\frac{dY_t - \overline{\mu}_t dt}{\sigma_Y^2}\right] + e^{-(r+\alpha+\gamma)(\tau'-\tau)}F_{\tau'}
\end{aligned}$$

is a martingale, as a function of τ'. Using the representation of the relational capital established above, the change of this martingale equals

$$\begin{aligned}
&(r+\alpha)\left[e^{-(r+\alpha+\gamma)(\tau'-\tau)}I_{\tau'} + \left((a_{\tau'} - c(a_{\tau'})) - rw_{\tau'}\right)\int_{\tau}^{\tau'} e^{-r(\tau'-t)}e^{-(r+\alpha+\gamma)(t-\tau)}\frac{dY_t - \overline{\mu}_t dt}{\sigma_Y^2}\right] \\
&\quad + \frac{d}{dt}e^{-(r+\alpha+\gamma)(\tau'-\tau)}F_{\tau'},
\end{aligned}$$

where the first term is the covariance of the Brownian increments of $(r+\alpha)w_{\tau'}$ and of the bracketed stochastic intergral in the last line. Integrating over $[\tau, \infty)$ and taking expectation at time τ yields the formula in (11).

Finally, since effort increases fundamentals by $(r+\alpha)dt$, and given the decomposition of the continuation value as in (7), the effort process is a local SSE exactly when a_t satisfies $a_t = a(F_t)$ (see e.g. the Verification Theorem in Yong and Zhou (1999) Ch.3.2). This establishes the proof.

Lemma 2 *Let $E : [\underline{w}, \overline{w}] \to R$ be a C^2 strictly concave function that satisfies the differential inequality*

$$(r+\alpha+\gamma)E(w) \le \max_I \left\{ (r+\alpha)I + E'(w)\left(rw - [a(E(w)) - c(a(E(w)))]\right) + \frac{E''(w)}{2}\sigma_Y^2 I^2 \right\}$$
$$= E'(w)\left(rw - [a(E(w)) - c(a(E(w)))]\right) - \frac{(r+\alpha)^2}{2\sigma_Y^2 E''(w)}, \tag{19}$$

where a is defined in (10), together with boundary conditions for each $w^\partial \in \{\underline{w}, \overline{w}\}$:

$$(w^\partial, E(w^\partial)) \in \mathcal{E}, \tag{20}$$

or,

$$(r+\alpha+\gamma)E(w^\partial) = E'(w^\partial)\left(rw^\partial - \left[a(E(w^\partial)) - c(a(E(w^\partial)))\right]\right), \tag{21}$$
$$sgn\left(\frac{\overline{w}+\underline{w}}{2} - w^\partial\right) = sgn\left(rw^\partial - \left[a(E(w^\partial)) - c(a(E(w^\partial)))\right]\right).$$

Then each point on the curve is achieved by a local SSE, $(w_0, F(w_0)) \in \mathcal{E}$ for $w_0 \in [\underline{w}, \overline{w}]$.

The result implies that any solution E of the HJB equation (12)—equation version of (19)—with boundary conditions (20-21) provides a lower bound for the supremum F of relational incentives achievable in a local SSE. The proof constructs a local SSE, based on the maximizer in (19), that achieves relational incentives $E(w)$. Hence, the result corresponds to one-half of a Verification Theorem.

In further detail, Ito's formula implies that if $\{w_t\}$ is the relational capital that follows (11) with policies $\{I_t\} = \{I^*(w_t)\}$, where $I^*(w)$ is the point-wise maximizer ("feedback control") and E is a solution to the HJB equation (12), then $\{E(w_t)\}$ is the associated process of relational incentives, as in Lemma 1. When relational capital reaches a boundary point that is known to be achievable by a local SSE, the game simply follows this local SSE. Under the alternative boundary conditions (21), relational capital is reflected back, and the construction as above continues.[38, 39]. The lemma is a convenient tool to find non-trivial, tractable—not necessarily optimal—local SSE (see, for example, the proof of Proposition 2)

Proof of Lemma 2. Let $I : [\underline{w}, \overline{w}] \to R$, with $I(w) \geq I^*(w) = -\frac{r+\alpha}{\sigma_Y^2 E''(w)}$, be the strictly positive continuous function that solves (19) with equality, i.e.,

$$(r+\alpha+\gamma)E(w) = (r+\alpha)I(w) + E'(w)\left(rw - [a(E(w)) - c(a(E(w)))]\right) + \frac{E''(w)}{2}\sigma_Y^2 I^2(w). \quad (22)$$

Note that the first boundary condition in (21) can hold either when $I\left(w^\partial\right) = 0$, or, as in our case, $I(w^\partial) = -2\frac{r+\alpha}{\sigma_Y^2 F''(w)} > 0$. The construction of a local SSE that achieves the boundary in the case $I(w^\partial) > 0$, when relational capital "escapes" the interval $[\underline{w}, \overline{w}]$, requires an additional step, as we detail below.[40]

First, we extend the functions E and I beyond the boundary points w^∂, at which condition (21) is satisfied with $I\left(w^\partial\right) > 0$ as follows. Consider a boundary point $w^\partial = \overline{w}$ and, say, $rw^\partial - \left(a(E(w^\partial)) - c(a(E(w^\partial)))\right) < 0$. We use the Implicit Function Theorem to extend function E to a point $\overline{\overline{w}} > \overline{w}$, so that conditions (21) and $E''(w) < 0$ hold on $\left[\overline{w}, \overline{\overline{w}}\right]$. We also extend I continuously to the interval $\left[\overline{w}, \overline{\overline{w}}\right]$ with $I(w) = -2\frac{r+\alpha}{\sigma_Y^2 E''(w)} > 0$, so that E and I satisfy the equation (22) on $\left[\overline{w}, \overline{\overline{w}}\right]$. In words, on the interval $\left[\overline{w}, \overline{\overline{w}}\right]$ the relational incentives can be provided in two ways: they can either consist entirely of the discounted future relational incentives, with zero flow, or by providing inefficiently high flow of relational incentives I. The extension to the interval $\left[\underline{\underline{w}}, \underline{w}\right]$ in the case of $w^\partial = \underline{w}$ is analogous.

Fix $w_0 \in [\underline{w}, \overline{w}]$. We first construct a process $\{w_t\}$ of continuation values that satisfies the stochastic equation (11). Let τ^∞ be the stopping time when $\{w_t\}$ reaches a boundary point that is a local SSE. Moreover, define a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}_+}$ such that $\tau_0 = 0$; for n odd, $\tau_n \geq \tau_{n-1}$ is the stopping time when $\{w_t\}$ reaches either of the new, "outside" boundary points $\left\{\underline{\underline{w}}, \overline{\overline{w}}\right\}$; and for $n > 0$ even, $\tau_n \geq \tau_{n-1}$ is the stopping

[38] More precisely, the stock of incentives $E\left(w^\partial\right)$ at the boundary can be generated by having either $I\left(w^\partial\right) = 0$, with relational capital drifting back inside of $[\underline{w}, \overline{w}]$, or $I(w^\partial) = -2\frac{r+\alpha}{\sigma_Y^2 E''(w)} > 0$. In the proof in Appendix A.3, we show how to reduce the second case to the first one by extending the functions E and I beyond $[\underline{w}, \overline{w}]$, with $I = 0$.

[39] When (19) is an inequality, the construction is analogous, but with a continuous feedback control function $I : [\underline{w}, \overline{w}] \to \mathbb{R}_+$, $I(w) \geq I^*(w)$, for which (19) holds with equality. The existence of such function I follows from strict concavity of E.

[40] We provide an elementary construction below. For the existence results of "Sticky Brownian Motion" based on the local time techniques see, e.g., Engelbert and Peskir (2014).

time when $\{w_t\}$ reaches either of the original "inside" boundary points $\{\underline{w}, \overline{w}\}$. For times $t \in [\tau_n, \tau_{n+1})$ with n even and $t < \tau^\infty$ we let $\{w_t\}$ be the weak solution to (11), with $I_t = I(w_t)$ and $\{M_t^w\} = 0$, starting at w_{τ_n}. Existence of a weak solution follows from the continuity of it's drift (which is a consequence of continuity of E and action defined via (10)) and volatility I (see e.g. Karatzas (1991), Theorem 5.4.22). For times $t \in [\tau_n, \tau_{n+1})$ with n odd and $t < \tau^\infty$ we let $\{w_t\}$ be the weak solution to (11), with $I_t = 0$ and $\{M_t^w\} = 0$, starting at w_{τ_n}. In words, the process $\{w_t\}$ has positive volatility until it reaches an "outside" boundary point in $\{\underline{\underline{w}}, \overline{\overline{w}}\}$, after which it drifts "inside" till it reaches the "inside" boundary point in $\{\underline{w}, \overline{w}\}$, when it resumes with the positive volatility, and so on.

It follows from Ito's formula that before τ^∞ the process $F_t = E(w_t)$, satisfies the equation in (11), with $J_t = E'(w_t) \times I(w_t)$. Since both w_t and F_t are bounded, the transversality conditions are satisfied. Finally, we may extend the processes $\{w_t\}, \{I_t\}, \{F_t\}$ and $\{J_t\}$, together with martingales $\{M_t^w\}$ beyond τ^∞ by letting them follow a local SSE that achieves $(w_{\tau^\infty}, F(w_{\tau^\infty}))$. Then the processes satisfy conditions of Lemma 1.

Lemma 3 *For any $\lambda > 0$, let $E^\lambda : [\underline{w}, \overline{w}] \to R$ be a concave function that satisfies the differential inequality*

$$(r+\alpha+\gamma)E^\lambda(w) \geq \max_I \left\{ (r+\alpha)I + E^{\lambda\prime}(w)\left(rw - \left[a(E^\lambda(w)) - c(a(E^\lambda(w)))\right]\right) + \frac{E^{\lambda\prime\prime}(w)\sigma_Y^2}{2} I^2 \right\} + \lambda$$
$$= E^{\lambda\prime}(w)\left(rw - \left[a(E^\lambda(w)) - c(a(E^\lambda(w)))\right]\right) - \frac{(r+\alpha)^2}{2\sigma_Y^2 E^{\lambda\prime\prime}(w)} + \lambda, \tag{23}$$

on an interval $[\underline{w}, \overline{w}]$, together with $\left|E^{\lambda\prime}\right| \leq 1/\lambda$ and the boundary conditions $E^\lambda(\underline{w}) = F(\underline{w})$ and $E^\lambda(\overline{w}) = F(\overline{w})$. Then there is $\delta := \delta(\lambda) > 0$ such that it is not possible that the boundary F satisfies

$$E^\lambda(w) < F(w) \leq E^\lambda(w) + \delta, \text{ for } w \in (\underline{w}, \overline{w}).$$

The lemma above establishes a novel escape argument and, hence, the second half of a Verification Theorem for our setting. If the law of motion of w_t did not depend on the level of the value function (relational incentives), as in a standard stochastic control problem, then the result would hold with $\lambda = 0$ and $\delta = \infty$. Any value above the solution of an HJB equation could be justified only by it drifting ever higher. In other words, a solution E of the HJB equation (12) with the given boundary conditions would provide an upper bound for the supremum F of relational incentives achievable in a local SSE.

In our setting, the level of the value function (relational incentives) affects the law of motion of the state variable, through the effort a_t. Thus, relational incentives higher than the solution to (12) may affect the effort and increase the right-hand side of the HJB equation. The lemma establishes only a “local” version of the bound: the boundary F of relational incentives cannot reach locally above (a local perturbation of) a solution E of the HJB equation, with given boundary conditions.

The proofs of Theorems 1 and 2 show how Lemmas 2 and 3 are sufficient to establish the characterizations. In particular, they establish that the boundaries F and F_ε are smooth and satisfy the HJB equation (12), and derive the respective boundary conditions.

Proof of Lemma 3. Fix an interval $[\underline{w}, \overline{w}]$ and a function $E^\lambda : [\underline{w}, \overline{w}] \to R$ that solves the differential inequality (23) as in the Lemma, as well as δ small (see below). Suppose, by contradiction, that the boundary F satisfies $E^\lambda(\underline{w}) = F(\underline{w})$, $E^\lambda(\overline{w}) = F(\overline{w})$, and $E^\lambda(w) < F(w) \leq E^\lambda(w) + \delta$, for $w \in (\underline{w}, \overline{w})$. Fix any (w_0, F_0) with $w_0 \in (\underline{w}, \overline{w})$ together with a local SSE that achieves it, and let $\{w_t\}$ and $\{F_t\}$ be the processes of relational capital and relational incentives it gives rise to. Define $D(w_t, F_t)$ as the distance of F_t from E^λ,

$$D(w_t, F_t) = F_t - E^\lambda(w_t).$$

Using Ito's lemma together with Lemma 1, at any time t when $D(w_t, F_t) \in [0, \delta]$, the drift of the process $D(w_t, F_t)$ equals, for appropriate process $\{I_t\}$,

$$\begin{aligned} \frac{\mathbb{E}\left[dD(w_t, F_t)\right]}{dt} &= (r+\alpha+\gamma) F_t - (r+\alpha) I_t - E^{\lambda\prime}(w_t) \times (r w_t - (a(F_t) - c(a(F_t)))) \\ &\quad - \frac{E^{\lambda\prime\prime}(w)\left[\sigma_Y^2 I_t^2 + d\langle M_t^w\rangle\right]}{2} \\ &\geq (r+\alpha+\gamma) F_t - (r+\alpha) I_t - E^{\lambda\prime}(w_t) \times \Big(r w_t - \Big(a(E^\lambda(w_t)) - c(a(E^\lambda(w_t)))\Big)\Big) \\ &\quad - \frac{E^{\lambda\prime\prime}(w)\left[\sigma_Y^2 I_t^2 + d\langle M_t^w\rangle\right]}{2} - \frac{\lambda}{2} \\ &\geq (r+\alpha+\gamma)\Big(F_t - E^\lambda(w_t)\Big) + \lambda - \frac{\lambda}{2} > (r+\alpha+\gamma) \times D(w_t, F_t), \end{aligned} \tag{24}$$

The first inequality holds because $\left|E^{\lambda\prime}(w_t)\right| \leq 1/\lambda$, functions a and c are Lipschitz continuous and $D(w_t, F_t) \in [0, \delta]$, where δ is assumed to be sufficiently small. The second inequality follows because E^λ satisfies

$$(r+\alpha+\gamma)E^\lambda(w) \geq \max_I \{(r+\alpha)I + E^{\lambda\prime}(w)\Big(rw - \Big(a(E^\lambda(w)) - c(a(E^\lambda(w)))\Big)\Big) + \frac{E^{\lambda\prime\prime}(w)\sigma_Y^2}{2}I^2\} + \lambda, \tag{25}$$

E^λ is concave, and $d\langle M_t^w\rangle$ is positive.

Let τ be the stopping time of the process $D(w_t, F_t)$ hitting zero. Due to $D(w_0, F_0) > 0$ and $\{D(w_t, F_t)\}_t$ being a submartingale, from (24), it follows that there is a finite time T such that $E[D(w_T, F_T)|\tau \geq T] > \delta$. Moreover,

$$\begin{aligned} 0 < D(w_0, F_0) \leq E\left[D(w_{\min\{T,\tau\}}, F_{\min\{T,\tau\}})\right] &= P(\tau \geq T) \times E[D(w_T, F_T)|\tau \geq T] \\ &+ P(\tau < T) \times E[D(w_\tau, F_\tau)|\tau < T] \\ &= P(\tau \geq T) \times E[D(w_T, F_T)|\tau \geq T], \end{aligned}$$

which implies $P(\tau \geq T) > 0$. This establishes that $D(w_T, F_T)$ exceeds δ with positive probability, contradiction.

A.2 Proof of Theorem 1

We begin the proof of the theorem with the following two technical lemmas. Recall that the efficient level of relational capital is $w_{EF} = \frac{1}{r}(a_{EF} - c(a_{EF}))$, for the efficient effort level a_{EF}, with $c'(a_{EF}) = 1$. Note that with the quadratic cost of effort as in (16), we have $a_{EF} = \frac{1}{2C}$, $a_{EF} - c(a_{EF}) = \frac{1}{8C}$, and $w_{EF} = \frac{1}{8Cr}$. Let also $\underline{F}$ be the lower arm of the parabola, which is the locus of the feasible relational capital-incentives pairs (w, F) that can be achieved by symmetric play in a stage game, satisfying $rw = a(\underline{F}) - c(a(\underline{F}))$; see Figure 1.

Lemma 4 *The set $\mathcal{E}$ is convex and $w^* \leq w_{EF}$. Moreover, the upper boundary F satisfies $F(w) \geq \underline{F}(w) > 0$, $w \in (0, w^*)$.*

Proof. Convexity is immediate from the possibility of public randomization, and the inequality $w \leq w_{EF}$ follows from the definitions. For the second part of the lemma, suppose by the way of contradiction that there exists w, $0 \leq w < w^*$, such that $F(w) < \underline{F}(w)$. Note that at w the slope of F is smaller than the slope of $\underline{F}$: otherwise, the repeated static Nash point $(0,0)$, belonging to the graph of the convex function $\underline{F}$, would not overlap with the convex set $\mathcal{E}$. This implies that F is bounded away below $\underline{F}$ to the right of w, and so in any local SSE the relational capital has drift bounded away above zero, as long as $w_t \geq w$ (see (11)). The possibility of escape of relational capital beyond w^* establishes the contradiction. ■

Lemma 5 *Let $f, e : [x, y) \to \mathbb{R}$ be two concave functions such that*

$$i)\ f \leq e,$$
$$ii)\ f(x) = e(x) \text{ and } f'_+(x) = e'_+(x),$$
$$iii)\ e''_+(x) \text{ exists.}$$

Then either $f''_+(x)$ exists and equals $e''_+(x)$ or there is a function g with $g(x) = f(x)$, $g'_+(x) = f'_+(x)$ and $g''_+(x) < e''_+(x)$ such that $f \leq g$ in a right neighborhood of x.

Proof. Suppose that $f''_+(x)$ does not exist or is not equal to $e''_+(x)$. From i) and ii), this means that there is a $\varepsilon > 0$ and a decreasing sequence $\{x_n\} \to x$ such that

$$f(x_n) \leq e(x) + e'_+(x) \times (x_n - x) + \left(e''_+(x) - \varepsilon\right) \times (x_n - x)^2 .$$

However, concavity of f implies that the above inequality holds not only for the sequence $\{x_n\}$ but in a right neighborhood of x. This implies the result, with $g(w) = e(w) - \varepsilon(w - x)^2$ in a right neighborhood of x. ■

The proof of Theorem 1 rests on the following four lemmas. Relying on Lemmas 2 and 3, as well as the above two lemmas, they establish that: (i) the boundary points $(w, F(w))$, for $w > 0$, may not be generated by solely deferred incentives from the future and require strictly positive volatility of relational capital, or flow of incentives (Lemma 6); (ii) the boundary F is differentiable (Lemma 7); (iii) given any boundary point $(w, F(w))$ and a tangent vector F', the solution of HJB equation (12) with those boundary conditions must locally lie weakly above the boundary F (Lemma 8), as well as (iv) weakly below the boundary F (Lemma 9).

The lemmas thus establish that in the range where the boundary $F(w)$ is strictly positive, it must satisfy the HJB equation (12). The proof is then concluded by establishing the boundary conditions (13).

Lemma 6 *If $(1, F')$ is a tangent vector at $(w_0, F(w_0))$, with $w_0 > 0$, then*

$$(r + \alpha + \gamma)F(w_0) > F' \times (rw_0 - [a(F(w_0)) - c(a(F(w_0)))]) . \tag{26}$$

Proof. Pick $(w_0, F(w_0))$ with $w_0 > 0$; it follows from Lemma (4) that $F(w_0) > 0$. If the drift term is zero, $rw_0 - (a(F(w_0)) - c(a(F(w_0)))) = 0$, then (26) holds. Suppose then that the drift is strictly negative, $rw_0 - (a(F(w_0)) - c(a(F(w_0)))) < 0$ (when the inequality is reversed the proof is analogous), and such that inequality (26) fails. Assume also that $(w_0, F(w_0))$ is achieved by a local SSE, as opposed to being a limit of local SSE pairs – an assumption that we relax at the end of the proof.

Let $\overline{F}' \geq F'$ be such that (26) holds with equality, with $\overline{F}'$ in place of F'. Consider the function E defined over $[w_0, w']$, where w' is in the right neighborhood of w_0, such that E satisfies (26) with equality, with initial condition $(E(w_0), E'(w_0)) = (F(w_0), \overline{F}')$, and such that $w - (a(E(w)) - c(a(E(w)))) < 0$ for all $w \in [w_0, w']$. E is the solution of the implicit function second order ordinary differential equation.

Since $(w_0, E(w_0))$ is achieved by a local SSE and the boundary condition (13) holds at w', the function E satisfies conditions of Lemma 2, together with $I \equiv 0$. Consequently, there are local SSE that achieve every pair in its graph, and so the function lies below the boundary, $E(w) \leq F(w)$, $w \in [w_0, w']$. Note that it follows that the strict inequality $(r + \alpha + \gamma)F(w_0) < F' \times (rw_0 - [a(F(w_0)) - c(a(F(w_0)))])$ is impossible, or else $\overline{F}' > F'$ and E lies above F.

Consider now a strictly concave quadratic function G^* defined in the right neighborhood of w_0 with $(G^*(w_0), G^{*\prime}(w_0)) = (F(w_0), F'(w_0)$ and $G^*(w) < E(w)$ for $w > w_0$. The function satisfies

$$(r + \alpha + \gamma) G^*(w) < G^{*\prime}(w) (rw - [a(G^*(w)) - c(a(G^*(w)))]) - \frac{(r+\alpha)^2}{2\sigma_Y^2 G^{*\prime\prime}}, \tag{27}$$

in a right neighborhood of w_0. But then, by increasing slightly $G^{*\prime}(w_0)$, we may construct a quadratic function G over an interval $[w_0, \overline{w}]$ that also satisfies (27), together with $G(w_0) = F(w_0)$, $G'(w_0) > F'(w_0)$, and $G(\overline{w}) < F(\overline{w})$. There exists then a function $I : [w_0, \overline{w}] \to \mathbb{R}$, with $I(w) > -\frac{(r+\alpha)^2}{\sigma_Y^2 G''}$, such that

$$(r + \alpha + \gamma) G(w) = I(w) + G'(w) (rw - [a(G(w)) - c(a(G(w)))]) + \frac{G'' \sigma_Y^2}{2} I^2. \; w \in [w_0, \overline{w}]$$

Applying Lemma 2, each point $(w, G(w))$, for $w \in [w_0, \overline{w}]$, can be achieved by a local SSE. Since $G'(w_0) > F'(w_0)$, this yields the desired contradiction.

Finally, when $(w_0, F(w_0))$ is not achieved by a local SSE, the result follows for the functions E, G^*, and G defined analogously as before, but with $E(w_0) = G^*(w_0) =$

$G(w_0) = F(w_0) - \varepsilon$, for sufficiently small $\varepsilon > 0$. ■

Consider now the HJB equation (12), written as $F''(w) = \mathcal{F}(w, F, F')$. Lemma 6 implies that the right hand side of this equation is well defined and is Lipschitz continuous in the neighborhood of the points $(w_0, F(w_0), F')$, for any w_0 in $(0, w^*)$ and a tangent vector $(1, F')$, with $F'' < 0$. The following corollary is used repeatedly in the proof of the theorem:

Corollary 4 *The solution of the HJB equation (12) exists and depends continuously on the initial parameters in the neighborhood of the boundary condition* $(w_0, F(w_0), F')$, *for any* w_0 *in* $(0, w^*)$ *and a tangent vector* $(1, F')$.

Lemma 7 *The upper boundary* F *of the set of relational capital and relational incentives achievable in a local SSE is differentiable in* $(0, w^*)$.

Proof. Suppose to the contrary that $(w_0, F(w_0))$ is a kink. If follows from Lemma 6 that for any tangent vector $(1, F')$ at $(w_0, F(w_0))$

$$(r + \alpha + \gamma)F(w_0) > F' \times (rw_0 - [a(F(w_0)) - c(a(F(w_0)))]) \, .$$

Continuous dependence on the initial parameters implies that there exists $\lambda > 0$ such that $E^{\lambda *}$ solving (23) with the same initial conditions is strictly above the curve F in a neighborhood of w_0 (excluding point w_0). Invoking the continuous dependence once again, this time shifting the initial condition $(w_0, F(w_0), F')$ down to $(w_0, F(w_0) - \varepsilon, F')$, for $0 < \varepsilon << \lambda$, we construct a function E^{λ} that satisfies the conditions of Lemma 3, yielding a contradiction. ■

Lemma 8 *For any* w_0 *in* $(0, w^*)$, *the solution* E *to the differential equation (12) with initial condition* $(w_0, F(w_0), F'(w_0))$ *is weakly above the curve* F *in a neighborhood of* w_0.

Proof. Suppose to the contrary that $E < F$ in, say, the right neighborhood of w_0 (the case of the left neighborhood is analogous). From continuous dependence on the initial parameters, there are $\varepsilon, \delta > 0$ such that that the solution $\widetilde{E}$ of (12) with initial conditions $(w_0, F(w_0) - \delta, F'(w_0) + \varepsilon)$ crosses above and then comes back to F, meaning $\widetilde{E}(w_1) > F(w_1)$ and $\widetilde{E}(w_2) < F(w_2)$ for some $w_2 > w_1 > w_0$. But then the function $\widetilde{E}$ defined on $[w_0, w_2]$ satisfies conditions of Lemma 2, and so its graph is achievable by local SSE. This yields a contradiction. ■

Lemma 9 *For any w_0 in $(0, w^*)$, the solution E to the differential equation (12) with initial condition $(w_0, F(w_0), F'(w_0))$ is weakly below the curve F in a neighborhood of w_0.*

Proof. Let E satisfy (12) with initial conditions $(w_0, F(w_0), F'(w_0))$ and suppose that either $F''_+(w_0)$ does not exist, or $F''_+(w_0) \neq E''_+(w_0)$ (the case of left second derivative is analogous). Lemmas 7 and 8 establish that the conditions of Lemma 5 are satisfied, with $x = w_0,\ e = E,\ f = F$, and so in the right neighborhood of w_0 F is bounded above by $E(w) - \bar{\varepsilon}(w - w_0)^2$, for appropriate $\bar{\varepsilon} > 0$. Continuous dependence on initial parameters implies that there exists $\varepsilon > 0$ such that $E^{\lambda *}$ solving (23) with the same initial conditions $(w_0, F(w_0), F'(w_0))$ as E has second derivative at w_0 strictly larger than $E''(w_0) - \bar{\varepsilon}$ and is strictly above the curve F in a right neighborhood of w_0 (excluding point w_0). Invoking the continuous dependence once again, this time turning the initial condition $(w_0, F(w_0), F'(w_0))$ right to $(w_0, F(w_0), F'(w_0) - \delta)$, for $0 < \delta << \lambda$, we construct a function E^λ that satisfies the conditions of Lemma 3, yielding a contradiction. ■

The proof so far established that the boundary F satisfies the HJB equation (12) on $(0, w^*)$. To conclude the proof of the theorem, it remains to establish the boundary conditions (13).

1. $F(0) = 0$. Strictly positive relational incentives at zero in a local SSE would imply that the expected discounted efforts by each agent are strictly positive; consequently, a deviation to zero effort always would yield a nonzero relational capital to a partner, contradiction.

2. $\lim_{w \uparrow w^*} F(w) = \underline{F}(w^*)$. i) Lemma 4 shows that $\lim_{w \uparrow w^*} F(w) < \underline{F}(w^*)$ is impossible. ii) If $\lim_{w \uparrow w^*} F(w) \in \big(\underline{F}(w_0), \overline{F}(w_0)\big)$, then, using Lemma 2, it would be possible to extend the solution to the right, with $I(w) = 0$ for $w > w^*$, contradiction. iii) If $\lim_{w \uparrow w^*} F(w) = \overline{F}(w_0)$ then, whether F approaches $\overline{F}$ from above or below, the differential equation (12) would be violated in the left neighborhood of w_*. iv) If $\lim_{w \uparrow w^*} F(w) > \overline{F}(w_0)$, then relational capital in any local SSE achieving points close to $(w^*, \lim_{w \uparrow w^*} F(w))$ has strictly positive drift, bounded away from zero. This would lead to the escape of w to the right of w^*, with positive probability, due to the strictly positive diffusion term.

3. $\lim_{w \uparrow w^*} F''(w) = -\infty$. When the condition is violated, then $I^*(w)$ is continuous and strictly positive close to w^*. The proof of the theorem so far establishes that F is C^2 and satisfies the differential equation (12). Given this regularity, standard verification theorem techniques establish that the equilibria achieving $(w, F(w))$, $w < w^*$, must use the optimal flow of relational incentives $I^*(w)$ a.e. (see Yong and Zhou (1999)); when $(w, F(w))$ is unattainable, the same is true for (w, E) in the limit, with E approaching $F(w)$. This, however, leads to the relational capital escaping to the right of w^*, with positive probability.

A.3 Proof of Theorem 2

In the proof, let $\overline{C}$ and $\underline{C}$ be the upper and the lower bounds on the second derivative of the cost function. In the following proofs we will need the following result.

Lemma 10 *For a function G, with $G = F$, as in Theorem 1, or $G = F_\varepsilon$, as in Theorem 2 and any $\varepsilon \geq 0$, we have*

$$G \leq \; 2 + \frac{(r+\alpha)}{8\underline{C}\sigma_Y r\sqrt{r+\alpha+\gamma}}. \tag{28}$$

Proof. Let $w^\# > 0$ be the argument where G attains maximum, $G'(w^\#) = 0$, and let $w^\circ > w^\#$ be such that $G(w^\circ) = \frac{1}{2}G(w^\#)$. Assume that $G(w^\#) > 2$ (or else the result is trivial). In this case, for all $w \in \left[w^\#, w^\circ\right]$

$$G(w) > 1 = \overline{F}(0) \geq \overline{F}(w),$$

and so the drift of relational capital $rw - (a(F(w)) - c(a(F(w))))$ is positive. Consequently,

$$(r+\alpha+\gamma)G(w) < -\frac{(r+\alpha)^2}{2\sigma_Y^2 G''(w)}, \quad w \in [w^\#, w^\circ]$$

or

$$-G''(w) < \frac{(r+\alpha)^2}{2\sigma_Y^2(r+\alpha+\gamma)G(w)} \leq \frac{(r+\alpha)^2}{\sigma_Y^2(r+\alpha+\gamma)G(w^\#)}, \quad w \in [w^\#, w^\circ]. \tag{29}$$

Summarizing, when $G(w^\#) > 2$ we have

$$\begin{aligned}\frac{1}{2}G(w^\#) = G(w^\#) - G(w^\circ) &< \frac{(r+\alpha)^2}{2\sigma_Y^2(r+\alpha+\gamma)G(w^\#)}(w^\circ - w^\#)^2 \\ &< \frac{(r+\alpha)^2}{2\sigma_Y^2(r+\alpha+\gamma)G(w^\#)}\left(\frac{1}{8Cr}\right)^2,\end{aligned}$$

where the first equality follows from the definition of w°, the first inequality follows from $G'(w^\#) = 0$ and the bound (29), and the last inequality follows from $w^\circ - w^\# < w_{EF} - 0 =$

$\frac{1}{8Cr}$. This concludes the proof of the lemma. ■

The proof of the first part of the theorem is analogous to the proof of Theorem 1. The optimal policy function implied by (14) is given by

$$
\begin{aligned}
I_\varepsilon^*(w) &= -\frac{r+\alpha}{\sigma_Y^2 F''(w)}, \quad \text{if } F_\varepsilon''(w) \geq -\frac{r+\alpha}{\sigma_Y^2 \varepsilon}, \\
I_\varepsilon^*(w) &= \varepsilon, \quad \text{if } -2\frac{r+\alpha}{\sigma_Y^2 \varepsilon} < F_\varepsilon''(w) < -\frac{r+\alpha}{\sigma_Y^2 \varepsilon}, \\
I_\varepsilon^*(w) &= 0, \quad \text{if } F_\varepsilon''(w) \leq -2\frac{r+\alpha}{\sigma_Y^2 \varepsilon}.
\end{aligned} \tag{30}
$$

In what follows we establish that if there is w such that $F_\varepsilon''(w) < -2\frac{r+\alpha}{\sigma_Y^2 \varepsilon}$, then $F_\varepsilon'(w)$ is negative and of order $\varepsilon^{-1/3}$. We claim that this is enough to establish the proof of the theorem. Indeed, we may define w_ε^* as the first point such that $F_\varepsilon''(w_\varepsilon^*) = -2\frac{r+\alpha}{\sigma_Y^2 \varepsilon}$. Note that, crucially, the policy I_ε^* is continuous and weakly above ε over $[0, w_\varepsilon^*)$, and so the constraint $I \notin (0, \varepsilon)$ is void. The existence of a local SSE then follows from the proof of Lemma 2. The bound in footnote 24 follows from concavity of F_ε, and the order of $F_\varepsilon'(w_\varepsilon^*)$.

Step 1. For w_ε^* defined as the first point such that $F_\varepsilon''(w_\varepsilon^*) = -2\frac{r+\alpha}{\sigma_Y^2 \varepsilon}$, we have

$$
\begin{aligned}
w_\varepsilon^* - w &= O(\varepsilon^{1/3}), \quad \text{when } F_\varepsilon' < 0, \\
w &= O\left(\varepsilon^{1/3}\right), \quad \text{when } F_\varepsilon' > 0.
\end{aligned}
$$

Consider w such that $F_\varepsilon''(w) < -2\frac{r+\alpha}{\sigma_Y^2 \varepsilon}$. Given quadratic costs, we have

$$
(a(F_\varepsilon(w)) - c(a(F_\varepsilon(w))))' = \frac{1}{C}\left(1/2 - F_\varepsilon(w)\right) F_\varepsilon'(w),
$$

for $C \in \left[\underline{C}, \overline{C}\right]$. Thus, differentiating (14), we get[41]

$$
\begin{aligned}
F_\varepsilon''(w) &= \frac{F_\varepsilon'(w)\left(\alpha + \gamma + [a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))]'\right)}{rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))} \\
&= \frac{F_\varepsilon'(w)\left(\alpha + \gamma + \frac{1}{C}\left(1/2 - F_\varepsilon(w)\right) F_\varepsilon'(w)\right)}{rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))} \\
&\geq -\frac{F_\varepsilon'^2(w)}{2\underline{C}|rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))|}, \quad \text{when } F_\varepsilon'(w) \leq 0 \\
&\geq -\frac{C_1 F_\varepsilon'^2(w)}{\underline{C}(rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w))))}, \quad \text{when } F_\varepsilon'(w) \geq 0
\end{aligned} \tag{31}
$$

[41]Note that (14) implies $F_\varepsilon'(w) \times [rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))] \geq 0$.

where C_1 is the bound on F_ε from Lemma 10. For an appropriate $C_2 > 0$ this yields

$$\frac{F_\varepsilon'^2(w)}{|rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))|} \geq \frac{C_2}{\varepsilon}. \tag{32}$$

On the other hand, equation (14) implies that

$$F_\varepsilon'(w)\left(rw - [a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))]\right) = (r + \alpha + \gamma)\, F_\varepsilon(w) \leq (r + \alpha + \gamma)\, C_1. \tag{33}$$

Inequalities (32) and (33) imply that $|rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))| \leq C_3 \varepsilon^{1/3}$, with $C_3 > 0$. Since $F_\varepsilon(w) \geq 1/2$, in the case when $F_\varepsilon'(w) \geq 0$ (so that the drift of relational capital is positive), whereas $F_\varepsilon(w) \geq \lim_{w \to w_\varepsilon^*} F_\varepsilon(w) = \underline{F}(w_\varepsilon^*) \geq C_4 > 0$, in the case when $F_\varepsilon'(w) \leq 0$ (the equality follows from the boundary condition (13)) equation (14) yields

$$|F_\varepsilon'(w)| = \frac{(r + \alpha + \gamma)\, F_\varepsilon(w)}{|rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))|} \geq C_5 \varepsilon^{-1/3}. \tag{34}$$

Since F_ε is concave, the inequality establishes Step 1.

Step 2. The case $F_\varepsilon'(w) \geq 0$ is not possible.

Since w is small and $rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))$ positive, we have $F_\varepsilon(w) \approx \overline{F}(0) = 1$. Differentiating (31),

$$\begin{aligned}
F_\varepsilon'''(w) &= \left(\frac{F_\varepsilon'(w)}{rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))}\right)' \left(\alpha + \gamma + (a(F_\varepsilon(w)) - c(a(F_\varepsilon(w))))'\right) \qquad (35)\\
&\quad + \frac{F_\varepsilon'(w)}{rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))} \left(a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))\right)'' \\
&\quad > \frac{F_\varepsilon'(w)}{rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))} \left(a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))\right)'' \\
&\quad =_{sgn} \left(a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))\right)'',
\end{aligned}$$

where the inequality follows from the fact that $F_\varepsilon''(w) < 0$ and

$$\begin{aligned}
(rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w))))' &= r - \frac{1}{C}\left(1/2 - F_\varepsilon(w)\right) F_\varepsilon'(w) \approx r + \frac{1}{2C} F_\varepsilon'(w) > 0, \\
\alpha + \gamma + (a(F_\varepsilon(w)) - c(a(F_\varepsilon(w))))' &= \alpha + \gamma + \frac{1}{C}\left(1/2 - F_\varepsilon(w)\right) F_\varepsilon'(w) \\
&\approx \alpha + \gamma - \frac{1}{2C} F_\varepsilon'(w) < 0,
\end{aligned}$$

for $C \in [\underline{C}, \overline{C}]$, when ε is small enough. Finally,

$$(a(F_\varepsilon(w)) - c(a(F_\varepsilon(w))))'' = \left(\frac{1}{C}\left(1/2 - F_\varepsilon(w)\right) F_\varepsilon'(w)\right)' \tag{36}$$

$$=_{sgn} \left(1/2 - F_\varepsilon(w)\right) F_\varepsilon''(w) - \left(F_\varepsilon'(w)\right)^2 \approx -\frac{1}{2} F_\varepsilon''(w) - \left(F_\varepsilon'(w)\right)^2$$

$$\approx \frac{1}{4C} \frac{\left(F_\varepsilon'(w)\right)^2}{rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))} - \left(F_\varepsilon'(w)\right)^2 > 0,$$

when ε is small enough, where the last line follows from (31). This establishes that $F_\varepsilon''(w^0) \leq -2\frac{r+\alpha}{\sigma_Y^2 \varepsilon}$ implies $F_\varepsilon'''(w^0) > 0$, and so the case $F_\varepsilon'(w^0) \geq 0$ is not possible. This establishes Step 2 and the proof of the Theorem.

Corollary 5 *On the interval* $[0, w_\varepsilon^*]$ *we have the bounds*

$$F_\varepsilon''(w) \geq 2\frac{r+\alpha}{\sigma_Y^2 \varepsilon},$$

$$F_\varepsilon'(w) \leq F_\varepsilon'(0) \leq \frac{1}{8r\underline{C}} \times 2\frac{r+\alpha}{\sigma_Y^2 \varepsilon}.$$

The first bound follows from the definition of w_ε^* as the first point such that $F_\varepsilon''(w_\varepsilon^*) = -2\frac{r+\alpha}{\sigma_Y^2 \varepsilon}$. The second line follows from $w_{EF} \leq 1/8r\underline{C}$ and $F_\varepsilon''(w) \geq 2\frac{r+\alpha}{\sigma_Y^2 \varepsilon}$ when $F_\varepsilon'(w) \geq 0$.

A.4 Proof of Theorem 3

Step 1. Fix $\varepsilon > 0$ and consider an $\varepsilon-$optimal local SSE $\{a_t, a_t\}$, together with the processes $\{w_t\}, \{F_t\}, \{I_t\}$ and $\{J_t\}$ that satisfy equations (11) (Proposition 1 and Theorem 2). In this step we show that as long as

$$J_t \leq \frac{C\left(r + 2\left(\alpha + \gamma\right)\right)}{8\left(r + \alpha\right)}, \quad \forall t \tag{37}$$

then, for an appropriate $X > 0$ and any deviating strategy $\{\widetilde{a}_t\}$, the relational capital at any time $\tau \geq 0$ to the deviating agent is bounded above by

$$\widetilde{w}_\tau(\widetilde{\mu}_\tau - \overline{\mu}_\tau, w_\tau) = w_\tau + \frac{F_\tau}{r+\alpha}(\widetilde{\mu}_\tau - \overline{\mu}_\tau) + X(\widetilde{\mu}_\tau - \overline{\mu}_\tau)^2. \tag{38}$$

In the formula, w_τ is the equilibrium level of relational capital, determined by (11), $\widetilde{\mu}_\tau$ are the correct beliefs, given strategies $\{\widetilde{a}_t\}$ and $\{a_t\}$, and $\overline{\mu}_\tau$ are the equilibrium beliefs, given that both strategies are $\{a_t\}$, both determined by (3). Consequently, using the bound with $\widetilde{\mu}_t = \overline{\mu}_t$, the step establishes that local SSE strategies are globally incentive compatible, as long as the bound (37) holds.

Fix a deviation strategy $\{\widetilde{a}_t\}$ and consider the process

$$v_\tau = \int_0^\tau e^{-rs}\left(\frac{\widetilde{a}_t + a_t}{2} - c(\widetilde{a}_t)\right)dt + e_\tau^{-r\tau}\widetilde{w}(\widetilde{\mu}_\tau - \overline{\mu}_\tau, w_\tau),$$

where, from (3), the wedge process $\{\widetilde{\mu}_t - \overline{\mu}_t\}$ follows

$$d\left(\widetilde{\mu}_t - \overline{\mu}_t\right) = (r+\alpha)\left(\widetilde{a}_t - a_t\right)dt - (\alpha+\gamma)\left(\widetilde{\mu}_t - \overline{\mu}_t\right)dt.$$

In order to establish that $\widetilde{w}_\tau$ bounds the relational capital under $\{\widetilde{a}_t\}$ and $\{a_t\}$, it is enough to show that the process $\{v_t\}$ has negative drift. We have

$$\begin{aligned}
e^{-rt}dv_t = &\left(\frac{\widetilde{a}_t + a_t}{2} - c(\widetilde{a}_t)\right)dt - r\left(w_t + \frac{F_t}{r+\alpha}(\widetilde{\mu}_t - \overline{\mu}_t) + X(\widetilde{\mu}_t - \overline{\mu}_t)^2\right)\\
&+ (rW_t - (a_t + c(a_t)))dt + I_t \times (dY_t - \overline{\mu}_t dt)\\
&+ \frac{\widetilde{\mu}_t - \overline{\mu}_t}{r+\alpha}\left((r+\alpha+\gamma)F_t - (r+\alpha)I_t dt + J_t \times (dY_t - \overline{\mu}_t dt)\right)\\
&+ \left(\frac{F_t}{r+\alpha} + 2X(\widetilde{\mu}_t - \overline{\mu}_t)\right)\left((r+\alpha)(\widetilde{a}_t - a_t)dt - (\alpha+\gamma)(\widetilde{\mu}_t - \overline{\mu}_t)dt\right).
\end{aligned}$$

Given that the drift of $dY_t - \overline{\mu}_t dt$ is $(\widetilde{\mu}_t - \overline{\mu}_t)dt$, the drift of the $e^{-rt}dv_t$ process equals

$$\begin{aligned}
&\frac{\widetilde{a}_t - a_t}{2} + c(a_t) - c(\widetilde{a}_t) + F_t(\widetilde{a}_t - a_t)\\
&+ (\widetilde{\mu}_t - \overline{\mu}_t)^2\left(\frac{J_t}{r+\alpha} - X\left(r + 2\left(\alpha+\gamma\right)\right)\right) + (\widetilde{\mu}_t - \overline{\mu}_t)(\widetilde{a}_t - a_t)2X\left(r+\alpha\right)\\
&\leq \frac{\widetilde{a}_t - a_t}{2} + c(a_t) - c(\widetilde{a}_t) + Ca_t(\widetilde{a}_t - a_t)\\
&+ (\widetilde{\mu}_t - \overline{\mu}_t)^2\left(\frac{J_t}{r+\alpha} - X\left(r + 2\left(\alpha+\gamma\right)\right)\right) + (\widetilde{\mu}_t - \overline{\mu}_t)(\widetilde{a}_t - a_t)2X\left(r+\alpha\right)\\
&= -\frac{C}{2}\left(a_t - \widetilde{a}_t\right)^2 + (\widetilde{\mu}_t - \overline{\mu}_t)^2\left(\frac{J_t}{r+\alpha} - X\left(r + 2\left(\alpha+\gamma\right)\right)\right)\\
&+ (\widetilde{\mu}_t - \overline{\mu}_t)(\widetilde{a}_t - a_t)2X\left(r+\alpha\right),
\end{aligned}$$

where we used that $c(a) = \frac{1}{2}a + \frac{C}{2}a^2$, and $F_t(\widetilde{a}_t - a_t) \leq Ca_t(\widetilde{a}_t - a_t)$, with equality in the case $a_t < A$.

Note that when the matrix

$$\begin{bmatrix} -\frac{C}{2} & X\left(r+\alpha\right) \\ X\left(r+\alpha\right) & \frac{J_t}{r+\alpha} - X\left(r + 2\left(\alpha+\gamma\right)\right) \end{bmatrix}$$

has a positive determinant, then the trace is negative, and the matrix is negative semidefinite, guaranteing negative drift. Since

$$\max_X \left\{ -\frac{C}{2} \times \left(\frac{J_t}{r+\alpha} - X\left(r + 2\left(\alpha+\gamma\right)\right) \right) - X^2 \left(r+\alpha\right)^2 \right\}$$
$$= \frac{C}{2\left(r+\alpha\right)} \left(\frac{C\left(r+2\left(\alpha+\gamma\right)\right)}{8\left(r+\alpha\right)} - J_t \right),$$

it follows that, indeed, when J_t is bounded as in (37), then $\widetilde{w}_\tau$ defined in (38) bounds the relational capital, for X that maximizes the above expression.

Step 2. Fix $\varepsilon > 0$ and consider an $\varepsilon-$optimal local SSE $\{a_t, a_t\}$. In this step we show that when $C\sigma_Y$ is sufficiently large, then for any w_t the sensitivity J_t of relational incentives is bounded as in (37). Together with step 1, this will establish the proof of Theorem 3.

Recall from Lemma 2 that

$$J_t = J(w_t) = F_\varepsilon'(w) \times I_\varepsilon^*(w).$$

Let us bound $I_\varepsilon^*(w)$, in the case when $F_\varepsilon'(w) > 0$. (Since $I_\varepsilon^* \geq 0$, the bound (37) holds in the case when $F_\varepsilon'(w) \leq 0$.) Over the subset $S \subseteq [0, \overline{w}_\varepsilon)$ where $F_\varepsilon''(w) < -\frac{r+\alpha}{\sigma_Y^2 \varepsilon}$, we simply have $I_\varepsilon^*(w) = \varepsilon$. Over the complement $[0, \overline{w}_\varepsilon)\backslash S$, where $F_\varepsilon''(w) \geq -\frac{r+\alpha}{\sigma_Y^2\varepsilon}$, we have,

$$\begin{aligned} I_\varepsilon^*(w) &= -\frac{r+\alpha}{\sigma_Y^2 F_\varepsilon''(w)} = \frac{2}{r+\alpha} \left\{ \left(r+\alpha+\gamma\right) F_\varepsilon(w) - F_\varepsilon'(w) \left(rw - \left(a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))\right)\right)\right\} \\ &\leq \frac{2}{r+\alpha} \left\{ \frac{(r+\alpha)\sqrt{r+\alpha+\gamma}}{8C\sigma_Y r} + 2(r+\gamma+\alpha) + \frac{r+\alpha}{4\sigma_Y^2 Cr\varepsilon}\frac{1}{8C} \right\}, \\ &= \frac{\sqrt{r+\alpha+\gamma}}{4C\sigma_Y r} + 4\frac{r+\gamma+\alpha}{r+\alpha} + \frac{1}{16C^2\sigma_Y^2 r\varepsilon} =: I^{\#} \end{aligned}$$

where we use the bound (28) on F_ε from Lemma 10, the bound on F_ε' from Corollary 5, and the lower bound of $-\left(a_{EF} - c\left(a_{EF}\right)\right) = -1/8C$ on the drift of relational capital.

Condition (37) thus boils down to

$$J_t = F_\varepsilon'(w) \times I_\varepsilon^*(w) \leq \frac{r+\alpha}{4\sigma_Y^2 Cr\varepsilon} \times I^{\#} \leq \frac{C\left(r+2\left(\alpha+\gamma\right)\right)}{8\left(r+\alpha\right)},$$

or,

$$\frac{\sqrt{r+\alpha+\gamma}}{4C\sigma_Y r} + 4\frac{r+\gamma+\alpha}{r+\alpha} + \frac{1}{16C^2\sigma_Y^2 r\varepsilon} \leq \frac{\left(r+2\left(\alpha+\gamma\right)\right)}{2\left(r+\alpha\right)^2} C^2 \sigma_Y^2 r \varepsilon, \tag{39}$$

which is satisfied when $C\sigma_Y$ is large enough. This concludes the proof of the step, and of the theorem.

A.5 Proofs for Section 4

Proof of Proposition 2. Part i) Recall that the efficient level of relational capital w_{EF} equals $\frac{1}{8Cr}$. The proof strategy is to construct a quadratic function E over the interval $[0,\overline{w}]$, with $\overline{w}=\frac{\delta}{8Cr}$ and fixed small δ (determined later) that satisfies the differential inequality

$$(r+\alpha+\gamma)E(w)\leq E'(w)\times(rw-[a(E(w))-c(a(E(w)))])-\frac{(r+\alpha)^2}{2\sigma_Y^2E''(w)},\qquad(40)$$

exactly as in Lemma 2, together with the left boundary condition $E(0)=0$ (achievable by the Markov equilibrium), and the right boundary condition (21). The result then follows from Lemma 2.

Specifically, given the quadratic cost of effort $c(a)=\frac{a}{2}+\frac{C}{2}a^2$, the flow payoffs (given interior efforts) satisfy

$$a(E)-c(a(E))=\frac{E(w)}{2C}\left(1-E(w)\right),$$

and also $\underline{F}'(0)=2Cr$. We let the right boundary condition

$$\begin{aligned}E(\overline{w})&=\frac{1}{2}>\underline{F}(\overline{w}),\\ E'(\overline{w})&=\frac{(r+\alpha+\gamma)E(\overline{w})}{r\overline{w}-\frac{E(\overline{w})}{2C}\left(1-E(\overline{w})\right)}=\frac{\frac{1}{2}(r+\alpha+\gamma)}{\frac{\delta}{8C}-\frac{1}{8C}}=\frac{-4C(r+\alpha+\gamma)}{1-\delta},\end{aligned}$$

so that the first equation in (21) is satisfied at $\overline{w}$; the second equation follows from $F(\overline{w})\in\left(\underline{F}(\overline{w}),\overline{F}(\overline{w})\right)$, which implies negative drift of relational capital. The constant second derivative D of function E is pinned down by

$$\begin{aligned}E\left(\overline{w}\right)&=\int_0^{\overline{w}}E'(x)dx=\int_0^{\overline{w}}\left[E'(\overline{w})-D\left(\overline{w}-x\right)\right]dx=E'(\overline{w})\times\frac{\delta}{8Cr}-\frac{D}{2}\left(\frac{\delta}{8Cr}\right)^2,\\ D&=2\left(E'(\overline{w})\frac{\delta}{8Cr}-E\left(\overline{w}\right)\right)\left(\frac{8Cr}{\delta}\right)^2=-\left(\frac{r+\alpha+\gamma}{r}\frac{\delta}{(1-\delta)}+1\right)\left(\frac{8Cr}{\delta}\right)^2\\ &\geq-\frac{128C^2r\left(r+\alpha+\gamma\right)}{\delta^2},\end{aligned}$$

with inequality holding as long as $\delta\leq\frac{1}{2}$.

We now establish the bounds for the function E. Let $w^{\#}$ be the point where E is maximized. For all $w \in [0, \overline{w}]$,

$$E(w) \leq E(w^{\#}) = \frac{1}{2} - \frac{D}{2}(\overline{w} - w^{\#})^2 = \frac{1}{2} - \frac{D}{2}\left(\frac{E'(\overline{w})}{D}\right)^2$$
$$= \frac{1}{2} + \frac{1}{2}\frac{16C^2\delta^2}{(1-\delta)^2 64C^2\left(\frac{r+\alpha+\gamma}{r}\frac{\delta}{(1-\delta)} + 1\right)}\frac{(r+\alpha+\gamma)^2}{r^2},$$
$$|E'(w)| \leq E'(0) \leq -D\overline{w} \leq \frac{128C^2\,(r+\alpha+\delta)}{\delta}.$$

Recall that the parabola, which is the locus of the feasible relational capital-incentives pairs (w, F) that can be achieved by symmetric play in a stage game, is defined via $rw = a(F) - c(a(F)) = \frac{F(w)}{2C}(1 - F(w))$ and, in particular, crosses the vertical axis at points $(0, 0)$ and $(0, 1)$ (see Figure 1). It thus follows from the bound on $E(w)$ above that when δ is small then the graph of the function E is within the parabola, for any values of parameters r, α, γ with a fixed ratio $\frac{r+\alpha+\gamma}{r}$. Consequently, for all $w \in [0, \overline{w}]$,

$$-\frac{1}{8C} \leq rw - \frac{E(w)}{2C}(1 - E(w)) \leq 0.$$

The last three bounds imply that when δ is small, then

$$(r+\alpha+\gamma)E(w) - E'(w)\left(rw - \frac{E(w)}{2C}\,(1 - E(w))\right) \leq (r+\alpha+\gamma)\left(1 + \frac{16C}{\delta}\right),$$
$$-\frac{(r+\alpha)^2}{2\sigma_Y^2 D} \geq \frac{(r+\alpha)^2}{r(r+\alpha+\gamma)}\frac{\delta^2}{256C^2\sigma_Y^2}.$$

The two inequalities establish that inequality (40) is satisfied, and so nontrivial local SSE exist, when $r + \alpha + \gamma$ is small enough so that

$$(r+\alpha+\gamma)\left(1 + \frac{16C}{\delta}\right) \leq \frac{(r+\alpha)^2}{r(r+\alpha+\gamma)}\frac{\delta^2}{256C^2\sigma_Y^2}. \tag{41}$$

To verify the existence of fully (globally) incentive compatible nontrivial SSE, note that the policy function $I\,(w)$ equals zero at the extremes, and for any $w \in (0, \overline{w})$ satisfies

$$I\,(w) \geq -\frac{r+\alpha}{\sigma_Y^2 D} \geq \frac{r+\alpha}{r(r+\alpha+\gamma)}\frac{\delta^2}{128C^2\sigma_Y^2} =: \varepsilon. \tag{42}$$

Condition (39) for global incentive compatibility, in the proof of Theorem 3, given (42), boils down to

$$\frac{\sqrt{r+\alpha+\gamma}}{4C\sigma_Y r} + 4\frac{r+\gamma+\alpha}{r+\alpha} + \frac{8}{\delta^2}\frac{r+\alpha+\gamma}{r+\alpha} \leq \frac{\delta^2}{256}\frac{r+2\,(\alpha+\gamma)}{(r+\alpha)^2}\frac{r+\alpha}{r+\alpha+\gamma}. \tag{43}$$

For a given ratio $\frac{r+\alpha+\gamma}{r}$, inequalities (41) and (43) hold when $r + \alpha + \gamma$ is sufficiently small. This concludes the proof of part i).

Results do not depend of normalizing the marginal benefit of effort: The proposition remains true when the effect of action is scaled up by $X > 1$, so that $d\mu_t = X(r+\alpha)(a_t^1 + a_t^2)dt - \alpha\mu_t dt + \sigma_\mu dB_t^\mu$ (for example, when the effect is independent of $r+\alpha$, we have $X = (r+\alpha)^{-1}$). We briefly comment here how the proof of the proposition must be adjusted.

For a fixed $X > 1$ the Markov equilibrium action becomes $a_M^X = \frac{X-1}{2C}$, and, given relational incentives F^X, the locally optimal action $a^X(F^X)$ equals $a_M^X + \frac{F^X}{C}$. The flow of relational capital (flow of equilibrium utility net of Markov equilibrium level) is $Xa^X(F^X) - c(a^X(F^X))$, which equals $\frac{F^X}{2C}\left(X - F^X\right)$; consequently, the HJB equation generalizes from (12) in Theorem 1 to

$$(r+\alpha+\gamma)F^X(w) = \max_I \left\{ X(r+\alpha)I + F^{X\prime}(w)\frac{F^X(w)}{2C}(X - F(w)) + \frac{F^{X\prime\prime}(w)\sigma_Y^2}{2}I^2 \right\} \quad (44)$$
$$= F^{X\prime}(w)\frac{F^X(w)}{2C}\left(X - F^X(w)\right) - \frac{X^2(r+\alpha)^2}{2\sigma_Y^2 F^{X\prime\prime}(w)}.$$

For the new parametrization, the construction remains analogous as in the proposition, with $a_{EF}^X = Xw_{EF}^1$, $w_{EF}^X = Xw_{EF}^1$, $\overline{w}^X = X\overline{w}^1$, $E^X(\overline{w}^X) = XE^1(\overline{w}^1)$, $E^{X\prime}(\overline{w}^X) = E^{1\prime}(\overline{w}^1)$, and $E^{X\prime\prime}(w) = \frac{1}{X}E^{1\prime\prime}(w)$. The bounds on function E in the proof change to: $\overline{E^X(w)} \leq X \times \overline{E^1(1)}$, $\overline{|E^{X\prime}(w)|} = \overline{|E^{1\prime}(w)|}$, and $0 \geq \underline{(rw - (2C^{-1})E^X(w)(X - E^X(w)))} \geq X \times \underline{(rw - (2C^{-1})E^X(w)(X - E^X(w)))}$. Consequently, all the terms in the inequality (40) are bounded by the terms scaled up by X, and the inequality continues to hold.

Part ii) Fix $\underline{w} = \delta \times w_{EF} = \frac{\delta}{8Cr}$, for some $\delta > 0$. We start with few preliminary observations. Suppose that $w^* > \underline{w}$. Observe that for all w such that $F'(w) \leq 0$ we have

$$F(w) \geq \lim_{s \to w^*} F(s) = \underline{F}(w^*) > \underline{F}(\underline{w}) \geq 2Cr\underline{w} =: A, \quad (45)$$

where the last inequality follows from $\underline{F}(0) = 0$, $\underline{F}'(0) = 2Cr$, and $\underline{F}$ convex. Secondly, recall from Theorem 1 that as w approaches w^* from the left, then $F'(w)$ gets arbitrarily high, and $F''(w)$ arbitrarily low. Finally, note that for any $w > 0$ the drift of the relational capital is uniformly bounded from below by

$$rw - (a(F(w)) - c(a(F(w)))) > -[a(F(w)) - c(a(F(w)))] \geq -[a_{EF} - c_{EF}] = -\frac{1}{8C} =: -B. \quad (46)$$

In the proof we establish that for the "discount factor" $r + \alpha + \gamma$ sufficiently high, if $w^* \geq \underline{w}$ and so $\underline{w}$ was achievable in a SSE, then the the value $F(w^{\#})$ of relational incentives at the point $w^{\#}$ such that $F'(w^{\#}) = 0$ would be arbitrarily high as well, for the function F characterized in Theorem 1. This will contradict the bound on $F(w^{\#})$ established in Lemma 10.

Fix $\overline{w}$ close to w^*, such that $-F''(\overline{w})$ equals $\varepsilon^{-1} > 0$ sufficiently large, to be determined later. Consider an auxiliary differential equation

$$(r+\alpha+\gamma)A = -G'(w)B - \frac{(r+\alpha)^2}{2\sigma_Y^2 G''(w)}, \tag{47}$$

together with a boundary condition $G(\overline{w}) = F(\overline{w})$, $G''(\overline{w}) = F''(\overline{w})$, and solved for $w \leq \overline{w}$.

Let $w^{\#\#} < \overline{w}$ be such that $G'\left(w^{\#\#}\right) = 0$. We argue that

$$G'(w) > F'(w), \text{ for all } w \in [w^{\#\#}, \overline{w}]. \tag{48}$$

Indeed, note that F satisfies equation (12), related to (47), but with $F(w)$ in place of A, and $rw - (a(F(w)) - c(a(F(w))))$, in place of $-B$. It follows from (45) and (46) that the inequality in (48) holds at $w = \overline{w}$. If (48) does not hold (in the entire range), then let $w^{\&}$, $w^{\#} < w^{\&} < \overline{w}$, be the maximal point such that $G'(w^{\&}) \leq F'\left(w^{\&}\right)$. It follows that $F(w^{\&}) > G(w^{\&})$ and $rw^{\&} - \left(a(F(w^{\&})) - c(a(F(w^{\&})))\right) > -B$, and so $G''(w^{\&}) < F''(w^{\&})$. This last inequality contradicts maximality of $w^{\&}$.

Inequality (48) implies that

$$F(w^{\#}) > G(w^{\#\#}), \tag{49}$$

for the maximal values of the respective functions, with $F'(w^{\#}) = 0$ and $G'(w^{\#\#}) = 0$.

We now compute $G(w^{\#\#})$. The solution to the differential equation (48) takes the form

$$G'(w) = \frac{\sqrt{\varepsilon^2 + 2c(\overline{w} - w)} - d}{c}, \quad G''(w) = -\frac{1}{\sqrt{\varepsilon^2 + 2c(\overline{w} - w)}},$$

for

$$d = 2\left(\frac{\sigma_Y}{r+\alpha}\right)^2 (r+\alpha+\gamma)\, A, \quad c = 2\left(\frac{\sigma_Y}{r+\alpha}\right)^2 B.$$

It follows that

$$
\begin{aligned}
w^{\#\#} &= \overline{w} - \frac{d^2-\varepsilon^2}{2c}, \\
G(w^{\#\#}) &= G(\overline{w}) - \int_{w^{\#\#}}^{\overline{w}} G'(w)dw = G(\overline{w}) - \int_{w^{\#\#}}^{\overline{w}} \frac{\sqrt{\varepsilon^2+2c(\overline{w}-w)}-d}{c}dw \\
&= G(\overline{w}) + \frac{d^2-\varepsilon^2}{2c}\frac{d}{c} + \frac{1}{c}\frac{2}{3}\frac{1}{2c}\left[\varepsilon^2+2c(\overline{w}-w)\right]^{3/2}\Big|_{w^{\#\#}}^{\overline{w}} \\
&= G(\overline{w}) + \frac{d^2-\varepsilon^2}{2c}\frac{d}{c} + \frac{\varepsilon^3}{3c^2}\frac{d^2-\varepsilon^2}{2c} - \frac{1}{3c^2}\left(d^2-\varepsilon^2\right)^{3/2} \geq \frac{d^3}{2c^2} - \frac{d^3}{3c^2} = \frac{1}{6}\frac{d^3}{c^2},
\end{aligned}
$$

where the last inequality holds when ε is chosen small enough. Substituting for d, c, B and A, with $\underline{w} = \frac{\delta}{8Cr}$, yields

$$
G(w^{\#\#}) \geq \frac{512}{3}\left(\frac{\sigma_Y}{r+\alpha}\right)^2 (r+\alpha+\gamma)^3 C^4 r^3 \left(\frac{\delta}{8Cr}\right)^3 = \frac{1}{3}\left(\frac{\sigma_Y}{r+\alpha}\right)^2 (r+\alpha+\gamma)^3 C\delta^3.
$$

Finally, given the above bound, the bound on F from Lemma 10 together with (49) imply

$$
\frac{1}{3}\left(\frac{\sigma_Y}{r+\alpha}\right)^2 (r+\alpha+\gamma)^3 C\delta^3 < 2 + \frac{(r+\alpha)}{8\underline{C}\sigma_Y r\sqrt{r+\alpha+\gamma}}. \tag{50}
$$

The inequality cannot be satisfied, when $r > 0$ and $r+\alpha+\gamma$ is sufficiently large. This concludes the proof of the proposition.

Results do not depend of normalizing the marginal benefit of effort: As in the case of part i), part ii) of the proposition remains true when the effect of action is scaled up by $X < 1$, so that $d\mu_t = X(r+\alpha)(a_t^1 + a_t^2)dt - \alpha\mu_t dt + \sigma_\mu dB_t^\mu$ (for example, when the effect is independent of $r+\alpha$, we have $X = (r+\alpha)^{-1}$). We briefly comment here how the proof of the proposition must be adjusted.

Fix $X < 1$; the bounds in the proposition change to $A^X = \frac{1}{X}A^1$, $B^X = X \times B^1$, and the last term in the equation (47) is scaled up by X^2 (see (44)). Consequently, $d^X = \frac{1}{X^3}d^1$, $c^X = \frac{1}{X} \times c^1$, $G^X(w^{X\#\#}) = \frac{1}{X^7}G^1(w^{1\#\#})$. This results in the left-hand side in the necessary inequality (50) multiplied by $X^{-7} > 1$.

Proof of Proposition 3. Part i) Suppose $\gamma = \sigma_\mu = 0$. We show that the supremum w_ε^* of relational capitals achievable in the $\varepsilon-$optimal local SSE is increasing in σ_Y^{-1}, for every $\varepsilon > 0$. Note that decreasing σ_Y changes equation (19) in Proposition 2 only by decreasing the last term. This means that if a pair of functions (F, I) satisfies the conditions of Lemma 2 for some interval $[\underline{w}, \overline{w}]$ and a given σ_Y, then for any σ_Y' with

$0 < \sigma_Y' < \sigma_Y$ there is a function $\overline{I} \geq I$ such that the pair $\left(F, \overline{I}\right)$ satisfies the conditions of Lemma 2 for σ_Y'. Applying the result to the pair $(F_\varepsilon, I_\varepsilon^*)$ on the interval $[0, \overline{w}_\varepsilon]$ as in the proof of Theorem 2, for any $\varepsilon > 0$, establishes the proof.

Part ii) Fix $\underline{w} = \frac{\delta}{8Cr}$, with $\delta > 0$. Proof of part ii) of Proposition 2 establishes that a necessary condition for $w^* \geq \underline{w}$ is inequality (50). Substituting for γ, the formula becomes

$$\frac{1}{3}\left(\frac{\sigma_Y}{r+\alpha}\right)^2 \left(r+\alpha+\sqrt{\alpha^2+\frac{\sigma_\mu^2}{\sigma_Y^2}}-\alpha\right)^3 C\delta^3 < 2 + \frac{(r+\alpha)}{8\underline{C}\sigma_Y r\sqrt{r+\alpha+\sqrt{\alpha^2+\frac{\sigma_\mu^2}{\sigma_Y^2}}-\alpha}}. \tag{51}$$

When σ_Y is close to zero, the left-hand-side is of order σ_Y^{-1}, whereas the right-hand-side is of order $\sigma_Y^{-1/2}$. This establishes that $w^* \leq \underline{w}$, when σ_Y is sufficiently small.

Part iii). As a preliminary step, we show that a symmetric strategy profile $\{a_t, a_t\}$ is an SSE with associated relational capital process $\{w_t\}$ if and only if there is an L^2 process $\{I_t\}$ such that

$$dw_t = (rw_t - (a_t - c(a_t)))\,dt + I_t \times (d\mu_t - [(r+\alpha)\,2a_t - \alpha\mu_t]\,dt) + dM_t^w, \tag{52}$$

where $a_t = a((r+\alpha)\,I_t)$, and $\{M_t^w\}$ is a martingale orthogonal to $\{Y_t\}$, and the transversality condition $\mathbb{E}\left[e^{-rt}w_t\right] \to_{t\to\infty} 0$ holds.

The proof is identical to the first part of the proof of Lemma 1: since the process $\left\{\mu_t - \int_0^t [(r+\alpha)\,2a_s - \alpha\mu_s]\,ds\right\}$, scaled by σ_μ, is a Brownian Motion, it follows from Proposition 3.4.15 in Karatzas (1991) that a process $\{w_t\}$ is the relational capital process associated with $\{a_t, a_t\}$, defined in (7), precisely when it can be represented as in (52), for some L^2 process $\{I_t\}$ and a martingale $\{M_t^w\}$ orthogonal to $\{\mu_t\}$.

As regards incentive compatibility, fix an alternative strategy $\{\widetilde{a}_t\}$ for player i and note that the relational capital satisfies

$$\begin{aligned}
&\mathbb{E}_\tau^{\{\widetilde{a}_t,a_t\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{\widetilde{a}_t+a_t}{2}-c(\widetilde{a}_t)\right)dt\right]\\
&=\mathbb{E}_\tau^{\{\widetilde{a}_t,a_t\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{\widetilde{a}_t+a_t}{2}-c(\widetilde{a}_t)\right)dt+w_\tau+\int_\tau^\infty d\left(e^{-rt}w_t\right)\right]\\
&=w_\tau+\mathbb{E}_\tau^{\{\widetilde{a}_t,a_t\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{\widetilde{a}_t+a_t}{2}-c(\widetilde{a}_t)\right)dt+\int_\tau^\infty e^{-rt}\left(dw_t-rw_tdt\right)\right]\\
&=w_\tau+\mathbb{E}_\tau^{\{\widetilde{a}_t,a_t\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{\widetilde{a}_t-a_t}{2}-c(\widetilde{a}_t)+c\left(a_t\right)+I_t\left(r+\alpha\right)\left(\widetilde{a}_t-a_t\right)\right)dt\right],
\end{aligned}$$

where the first equality follows from $\mathbb{E}_\tau^{\left\{\widetilde{a}_t^i,a_t^{-i}\right\}}\left[e^{-r(t-\tau)}w_t\right]\to 0$, as $t\to\infty$ (given that efforts are bounded), and the last one follows from $\mathbb{E}_\tau^{\{\widetilde{a}_t,a_t\}}\left[d\mu_t-\left[(r+\alpha)\,2a_t-\alpha\mu_t\right]dt\right]=(r+\alpha)\,\mathbb{E}_\tau^{\{\widetilde{a}_t,a_t\}}\left[\widetilde{a}_t-a_t\right]$. Since continuation value and relational capital differ by a constant, it follows from this representation and convexity of costs that there exists no profitable deviating strategy for partner i if and only if her effort process satisfies $a_t=a((r+\alpha)\,I_t)$.

We are now ready to establish part iii) of the proposition. From representation (52) it follows that when $w_t\geq\varepsilon>0$, then either the volatility satisfies $I_t\sigma_\mu\geq\delta>0$, in order to incentivize a strictly positive, more efficient effort, or the drift satisfies $\mathbb{E}_\tau^{\left\{a_t^1,a_t^2\right\}}\left[dw_t\right]\geq\delta>0$, to satisfy promise keeping (where δ depends on ε). It follows that if $w_0>0$ then the process $\{w_t\}$ escapes to infinity with positive probability, which, given bounded efforts, yields contradiction.

Proof of Proposition 4. Fix $\sigma_\mu>\sigma_\mu^\#\geq 0$; we show that, for any $\varepsilon>0$, the corresponding suprema of relational capitals achievable in the $\varepsilon-$optimal local SSE satisfy $w_\varepsilon^{\#*}\geq w_\varepsilon^*$. The proof is very related to the proof of Proposition 3. One extra complication is that now, changing the noise of the fundamentals also affects the boundary conditions (21) in Lemma 2, via the effect on γ.

Specifically, note that decreasing σ_μ changes equation (19) in Proposition 2 only by decreasing γ in the first term. Let $\gamma^\#\leq\gamma$ be the two corresponding gain parameters, and let $\overline{w}_\varepsilon$ be the relational capital achievable in a $\varepsilon-$optimal local SSE with σ_μ, as in the proof of Theorem 2, together with a pair of functions $(F_\varepsilon,I_\varepsilon^*)$ defined on $[0,\overline{w}_\varepsilon]$. Let $\left(F_\varepsilon^\#,I_\varepsilon^{\#*}\right)$ extend the functions $(F_\varepsilon,I_\varepsilon^*)$ to the right by letting $F_\varepsilon^{\#''}(w)=F_\varepsilon^{\#''}(\overline{w}_\varepsilon)$ and $I_\varepsilon^{\#*}(w)=I_\varepsilon^{\#*}(\overline{w}_\varepsilon)$, for $w>\overline{w}_\varepsilon$, and let $\overline{w}_\varepsilon^\#$ be the first argument such that the boundary condition (21) is satisfied. The existence of such $\overline{w}_\varepsilon^\#$ follows from the fact that at $\overline{w}_\varepsilon$

condition (21) is violated, with the left-hand-side too small (due to $\gamma^{\#} \leq \gamma$), and when w increases and $F_{\varepsilon}^{\#}$ decreases and approaches from above the value $\underline{F}(w)$ at which the drift dies out, the left-hand-side is bounded away from zero, and the right-hand-side converges to zero. It follows that for small $\sigma_{\mu}^{\#}$ and $\gamma^{\#}$, the pair $\left(F_{\varepsilon}^{\#}, I_{\varepsilon}^{\#*}\right)$ satisfies conditions of Lemma 2. This establishes the proof.