Partnership with Persistence

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Abstract

We study a continuous-time model of partnership with persistence and imperfect state monitoring. Partners exert private efforts to shape the stock of fundamentals, which drives the profits of the partnership, and the profits are the only public signal. The optimal strongly symmetric equilibria are characterized by a novel differential equation that describes the supremum of equilibrium incentives for any level of relational capital. Under (almost) perfect monitoring of the fundamentals, the only equilibria are (approximately) stationary Markov. Imperfect monitoring helps sustain relational incentives and increases the partnership's value by extending the relevant time horizon for incentive provision. The results are consistent with the predominance of partnerships and relational incentives in environments where effort has long-term and qualitative impact and in which progress is hard to measure.

Keywords: partnership, dynamic games, continuous time, relational incentives

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1 Introduction

Partnerships are among the main forms of organizing economic activity. Characterized by joint ownership, partnerships are common among individuals and businesses and constitute one of the dominant forms of structuring a firm—along with corporations and sole proprietorships.[1] Furthermore, partnerships embody the incentive problem of motivating members to exert private effort and contribute to the common good, which is common to many organizations.

The ongoing, dynamic nature of joint ownership complicates the incentive problem in a partnership. As an example, consider a start-up. On a daily basis, each partner devotes effort to improving the venture's fundamentals: upgrading the quality of the product; broadening the customer base; facilitating access to external capital; improving the internal organization; and more. Each of these fundamentals evolves over time, affected by the partners' efforts and by the circumstances. Moreover, none of them needs to be directly observed by the partners, who see only how the fundamentals are gradually reflected in the shared profits, customer reviews, or internal audits. At the same time, the ongoing nature of joint ownership offers unique opportunities to solve the incentive problem: it fosters relational incentives. A partner has incentives to work hard not only to boost profits but also to boost observable outcomes, morale, and, ultimately, to increase the future effort choices of the partners.

In this paper, we analyze incentives in a continuous-time model of partnership, with a persistent, stochastic state—the fundamentals—that is imperfectly monitored by the partners. Our first main contribution is to develop a method to solve the optimal strongly symmetric equilibria in such games. We provide a novel ordinary differential equation that characterizes the upper boundary of equilibrium incentives and the supremum of partnership values. Our second main contribution is to show that imperfect state monitoring

[1] According to the IRS data, in 2015 partnerships made up over 10% of all U.S. businesses, and accounted for over 25% of total net business income; see `https://www.irs.gov/statistics/soi-tax-stats-integrated-business-data`, Table 1. More broadly, teamwork, which shares the central feature of collective rewards and free-riding, was utilized in close to 80% of U.S. businesses at the turn of the century; see Lazear and Shaw (2007).

may increase equilibrium payoffs. If fundamentals are perfectly monitored, the partnership cannot provide relational incentives and relies on the unique Markov equilibrium—or nearly so when fundamentals are monitored closely. Imperfect monitoring benefits partners, in that it extends the time horizon for incentive provision, allowing future relational rewards to motivate today's effort. The results are consistent with the predominance of partnerships and relational incentives in environments in which effort has long-term and qualitative impact and in which progress is hard to measure.

In our continuous-time model, at any point in time, two partners privately exert costly effort and evenly split the profits of their venture. Fundamentals change stochastically—driven by the sum of efforts—and, in turn, equal the expected profit flow. In the model, neither efforts nor fundamentals are observable and profits, which follow a Brownian diffusion, are the partners' only publicly available information. Our minimal monitoring structure does not allow the signals to separately identify each partner's effort (Fudenberg, Levine, and Maskin (1994)) and, consequently, we focus on the strongly symmetric equilibria (SSE).

Partner's effort increases the fundamentals of the partnership and, thus, profits in the future. This benefits her in two distinct ways. First, she benefits directly by capturing half of the increased profits. In our model, those *Markov incentives* are constant, resulting in a unique, stationary Markov equilibrium (Proposition 1). Second, the increased profits affect the partners' effort decisions, such as when partners coordinate on relatively efficient (inefficient) efforts after surprisingly high (low) profit realizations, indicative of high (low) past efforts. The resulting *relational incentives* are our key focus.

The results in this paper rely on the persistent effect of effort and imperfect state monitoring, which together lengthen the time horizon for incentive provision, in the following sense. If the current profits depend only on current efforts (i.e., in a standard i.i.d. repeated game) or if fundamentals are perfectly monitored (i.e., in a standard stochastic game) then the rewards must be provided instantaneously. This is because an increased effort brings about unexpectedly good news (high profits or increase of fundamentals) only in the same period. Outside of those limiting environments, signals indicating increased effort today are spread over time. Rewards awarded for unexpectedly high profits in the future provide incentives to exert effort today. This change has a dramatic impact on the provision of relational incentives, as we discuss below.

First of all, poorer monitoring and the resulting increased time horizon for incentive provision may improve relational incentives and, thus, benefit the partnership. The reason is that relational rewards must take a form of a promise of an improved future relationship. If the relationship is already at the bliss point, then immediate relational rewards are unavailable.[2] We show that when monitoring is (nearly) perfect and hence the horizon for incentives is short, the absence of relational rewards at the bliss point of the relationship (nearly) unravels the provision of incentives (Proposition 3). In contrast, with poor monitoring, relational incentives do not unravel. In the absence of immediate rewards, partners are incentivized to work at the bliss point by future rewards, accruing once the relationship drifts down. The impossibility of relational incentives (Sannikov and Skrzypacz (2007, 2010)) may be attributed to the assumption of perfect state monitoring, and frequent moves or continuous-time modelling amplify its effect.

Second of all, the possibility of nontrivial relational incentives requires a novel method to characterize the optimal incentive provision. Unlike in repeated games, the optimal relational incentives in our setting are truly dynamic (in contrast to the "bang-bang" result in Abreu, Pearce, and Stacchetti (1986)). The method we propose is based on characterizing the upper boundary of relational incentives achievable in a local SSE, under local incentive constraints, as a function of the expected value of future efforts (relational capital, an equivalent of continuation value in an i.i.d. setting). Theorem 1 shows that the upper boundary of incentives satisfies an appropriate ordinary differential equation and provides boundary conditions. The right-most argument characterizes the supremum of relational capital and partnership's value. Theorem 2 shows, roughly, that a modified boundary is self-generating (as in (Abreu, Pearce, and Stacchetti, 1990)) and defines a near-optimal local SSE.

[2]Indeed, in a Brownian diffusion model like ours, no immediate incentives can be provided at the bliss point. Sannikov and Skrzypacz (2007) show that the impossibility persists in discrete-time models with short period lengths.

In our setting, with incentives as the value function, the law of motion of the state variable (relational capital) depends, via effort chosen, on the level of the value function. While the dependence is not allowed in standard stochastic control, we verify that the characterization of the boundary in the form of an HJB equation is valid.[3] Another difficulty is familiar: In Theorem 3, we provide conditions on the primitives so that the constructed strategies are not only locally, but fully incentive-compatible. We also provide a general model and the HJB characterization (Theorem 4) and discuss application to models of capital accumulation and oligopoly (Section 5).

Section 4 emphasizes the features of equilibrium dynamics and the effects on information structure specific to relational incentives. Following good outcomes, partners increase their efforts when relational capital is low, and decrease efforts when relational capital is high ("rallying and coasting"). Relational incentives may increase when effect of effort is more persistent. Improved monitoring may decrease partnership value, whereas less stochastic fundamentals, or less uncertainty about the quality of the partnership is always beneficial. This contrasts with the effects of information quality on career concerns or on reputational incentives. Partnerships thus have an edge in environments that favor established ventures, and in which the effects of effort are hard to measure or quantify (such as in the professional sector, Levin and Tadelis (2005)). Finally, an established partnership may unravel as a consequence of a short spat of bad outcomes, with hardly any effect on its expected fundamentals ("Beatles' break-up".)

1.1 Related Literature

This paper belongs to the literature on free-riding in groups, in dynamic environments.[4] The repeated partnership game was first studied by Radner (1985) and Radner, Myerson,

[3]We are aware of abusing the terminology. Formally, the optimality equation is not an HJB equation, as the state variable depends on the objective function. However, the optimality equation has the exact form of an HJB equation, with incentives F equal to the point-wise maximum, over all policies, of the expected flow of incentives, plus the stochastic differential operator applied to function F. We believe that the results are best viewed as extending stochastic control methods to allow for such dependency, parallel to the results in Sannikov (2007) and Faingold and Sannikov (2020).

[4]See Olson (1971), Alchian and Demsetz (1972), Holmstrom (1982), as well as Legros and Matthews (1993) and Winter (2004) for seminal contributions in static settings.

and Maskin (1986), who demonstrate inefficiency of equilibria, and by Fudenberg, Levine, and Maskin (1994), who pin down the identifiability conditions violated in the model; Abreu, Pearce, and Stacchetti (1986) show that symmetric equilibria feature a "bang-bang" property.

Our results provide a novel rationale for the impossibility of relational incentives found in the literature. Abreu, Milgrom, and Pearce (1991) and Sannikov and Skrzypacz (2007, 2010) show how frequent interactions may have a detrimental effect on incentives. In particular, the discrete-time approximation of a Brownian model of partnership or collusion in Sannikov and Skrzypacz (2007), which has either no persistence or a perfectly monitored state, has no relational incentives.[5] Faingold and Sannikov (2011) and Bohren (2018) establish related results with one long-lived player in a competitive market setting. We show that the impossibility is not inherent to continuous-time modeling but it is a consequence of the monitoring structure instead. Moreover, our results complement the discussion in Kandori and Obara (2006) and Rahman (2014), which show how relational incentives may be restored using private strategies and communication.

Similarly, a number of papers establish that, broadly speaking, better information may have a detrimental effect on a firm. In a model of non-contractible performance measures, better monitoring can exacerbate the principal's exploitation motive (see Zhu (2023)). Lizzeri, Meyer, and Persico (2002) and Fuchs (2007) show the benefits of limiting the times of feedback to the agent,[6] whereas in Cetemen, Hwang, and Kaya (2020) limited feedback may help partners by mitigating the ratchet effect. Similarly, in a linear Gaussian rating model of Hörner and Lambert (2021), Bonatti and Cisternas (2020) show that putting relatively much weight on old signals about a customer may mitigate the ratchet effect and benefit the firm (see also Ball (2022)).[7] Our paper identifies the negative effect of better monitoring on relational incentives, due to the shorter time horizon for incentives.

[5] See, also, Sadzik and Stacchetti (2015) for the discrete-time approximation of the Brownian Principal-Agent, rather than partnership model.

[6] In both models, improved monitoring, rather than limiting times when the feedback is observed, would lead to stronger terminal incentives and more efficient contracts.

[7] Quick learning and ratchet effect also prevent nontrivial effort in Bhaskar (2014), in a model that combines continuous and discrete choices.

Our paper ties into the literatures on career concerns (see Holmström (1999) and Cisternas (2017)) and on experimentation in teams (see Bolton and Harris (1999), Georgiadis (2014), and Cetemen, Hwang, and Kaya (2020) for Brownian models like ours).[8] In career concerns models, equilibrium play depends only on beliefs about an exogenous state;[9] the literature on experimentation in teams studies the effects of payoff or information externalities on incentives and focuses on Markov equilibria as well. Our paper is complementary: It has production technology independent of history, with constant Markov incentives, but we focus on optimal equilibria, which rely on relational incentives.[10] Hörner, Klein, and Rady (2022) investigate relational incentives in the experimentation in teams model.

Regarding our methodological contribution, the literature on dynamic contracting with persistence has long recognized the difficulty of accounting for incentives and of verifying global incentive compatibility (see Jarque (2010), Williams (2011), Prat and Jovanovic (2014), Sannikov (2014), Prat (2015), DeMarzo and Sannikov (2016), and He et al. (2017) for Brownian models like ours). Unlike in this literature, in our game setting, incentives are not treated as a state variable but as a maximized objective. Our HJB characterization, which allows for the evolution of the state variables to depend on the level of the value function, is related to the HJB characterization in Sannikov (2007), which allows the dependence on the slope of the value function, in a repeated game setting (see, also, Faingold and Sannikov (2020)). Moreover, we provide conditions on the primitives of the model for the global incentive compatibility, guaranteeing that the solution of the relaxed problem is fully incentive-compatible (see Williams (2011), Edmans et al. (2012), and Cisternas (2017) for related results).[11]

[8]See, among others, Keller, Rady, and Cripps (2005), Keller and Rady (2010), Klein and Rady (2011), and Bonatti and Hörner (2011) for experimentation in teams with exponential bandit models. See, also, Décamps and Mariotti (2004), Rosenberg, Solan, and Vieille (2007), Murto and Välimäki (2011), and Hopenhayn and Squintani (2011) for related stopping games with incomplete information.

[9]Specifically, Cisternas (2017) provides a differential equation also for the stock of incentives, just as in this paper, but in a differentiable Markov equilibrium, as a function of public beliefs about the state.

[10]In our near-optimal equilibria, working to rally the partnership is related to the encouragement effect identified by Bolton and Harris (1999), and coasting is reminiscent of the work-shirk-work dynamics in the reputation model of Board and Meyer-ter Vehn (2013).

[11]See, also, Sannikov (2014), and Prat (2015), who provide analytical conditions on the solution of the relaxed problem, under which the first-order approach is valid.

Finally, we interpret our results as providing a rationale for the prevalence of partnerships in industries with poor monitoring of the venture's progress. In particular, as documented by Von Nordenflycht (2010), "opaque" quality is a key characteristic of the knowledge-intensive environment of the professional sector, where partnerships are prevalent.[12] Our results are related to Levin and Tadelis (2005), who rely on partnership's comparative advantage in industries where employee quality is hard to evaluate, and to Morrison and Wilhelm (2004), who focus on partnership's impact on fostering mentorship relations.

2 Framework

2.1 Model

Two partners, who are risk-neutral and discount the future at a rate $r > 0$, play the following infinite horizon game: At every moment in time, $t \geq 0$, each partner i chooses effort a_t^i from an interval $[0, A]$.[13] Time t total effort contributes to the stock of fundamentals of the partnership, μ_t, which depreciates over time at a constant rate $\alpha > 0$. At any point in time, the stock of fundamentals is the mean of the partnership flow profits dY_t,

$$\begin{aligned} d\mu_t &= (r+\alpha)\,(a_t^1 + a_t^2)dt - \alpha\mu_t dt + \sigma_\mu dB_t^\mu, \\ dY_t &= \mu_t dt + \sigma_Y dB_t^Y, \end{aligned} \tag{1}$$

where $\{B_t^\mu\}$ and $\{B_t^Y\}$ are two independent Brownian Motions.[14] The multiplicative constant, $r + \alpha$, normalizes the total productivity of effort to one, regardless of the depreciation rate of the fundamentals or of the discount rate.[15] Finally, profits are the

[12]See Empson (2001) and Broschak (2004) for further references.

[13]The upper bound on the effort is used to guarantee boundedness of continuation value in Propositions 1 and 3, part iii). In all simulations, as well as in Theorem 2, the bound A is large enough so that equilibrium efforts are interior. Lemma 10 in Appendix A.3 bounds the relational incentives and, so, the efforts in a near-optimal equilibrium.

[14]Unless specified explicitly, all processes in this paper are indexed by $t \geq 0$.

[15]The constant is analogous to $1 - \delta$, which scales the stage game payoffs in repeated game analysis, where δ is the discount factor. The only results in which the normalization plays a role are the compar-

only publicly observable signal.

Exerting effort a entails a private flow cost $c(a)$, where $c(\cdot)$ is a twice differentiable, strictly convex cost of effort function. We normalize $c(0) = 0$ and $c'(0) = \frac{1}{2}$ (see Proposition 1), and in some of the results we further restrict the cost function to be quadratic (see Sections 3.2 and 4). Finally, at each point in time, partners split the profits evenly. Thus, given effort choices of both partners, a player's continuation payoffs are given by

$$W_\tau^i = \mathbb{E}_\tau^{\{a_t^1, a_t^2\}} \left[\int_\tau^\infty e^{-r(t-\tau)} \left(\frac{\mu_t}{2} - c(a_t^i) \right) dt \right].$$

Our partnership game has three features that extend the classic repeated-game framework: effort has persistent effect, state is imperfectly monitored, and partners learn about the fundamentals. Specifically, fundamentals, which are the state variable in the game, change only gradually over time driven by the efforts of the partners. Persistence of fundamentals implies that actions have a persistent effect: total effort today adds to the fundamentals, and, thus, also to the profit flow at any later time,

$$\mu_\tau = e^{-\alpha\tau}\mu_0 + \int_0^\tau e^{-\alpha(\tau-t)}(r+\alpha)\left(a_t^1 + a_t^2\right) dt + \sigma_\mu B_\tau^\mu.$$

Secondly, fundamentals need not be observed by the partners, who observe only noisy profit signals. Together with persistence, this implies that all future profits are useful signals of current efforts (see Proposition 1). Thirdly, fundamentals need not be determined by the efforts, and are changing stochastically. Alternatively, fundamentals are a sum of two terms: one that depends entirely on the past efforts of the partners, and the other that is purely stochastic, and reflects an unknown quality of the partnership. Consequently, in equilibrium partners do not know and keep on learning about the fundamentals.

The three features are parametrized in the model by α, $\sigma_Y, \sigma_\mu \geq 0$.[16] Their impact on the incentive provision in a partnership is one of the central themes of the paper, and

ative statics in Propositions 2, in which we show that equilibria with a nontrivial level of effort exist as $r+\alpha+\gamma$ converges to zero—even as the marginal benefit of effort on fundamentals, $r+\alpha$, disappears—and no effort is exerted in any equilibrium as $r+\alpha+\gamma$ converges to infinity—even as the effect of effort on fundamentals gets arbitrarily high. Both results continue to hold without the normalization, as we verify at the end of each proof, in Appendix A.5.

[16]We require that either σ_Y or σ_μ is strictly positive, to avoid the familiar complications in defining a continuous-time strategy in a game with perfect monitoring.

we discuss it at length in the following sections.

Public Beliefs Let $\overline{\mu}_\tau = \mathbb{E}_\tau^{\{a_t^1, a_t^2\}}[\mu_\tau]$ denote the public expected level of fundamentals at time τ, given efforts $\{a_t^1, a_t^2\}$. A simple application of the Kalman-Bucy filter yields that $\overline{\mu}_t$ follows

$$d\overline{\mu}_t = (r+\alpha)\left(a_t^1 + a_t^2\right)dt - \alpha\overline{\mu}_t dt + \gamma_t[dY_t - \overline{\mu}_t dt], \tag{2}$$

for an appropriate gain parameter γ_t, $dY_t = \overline{\mu}_t dt + \sigma_Y dB_t$, and a Brownian Motion $\{B_t\}$. We assume that, initially, partners believe that μ_0 is Normally distributed with steady-state variance σ^2. This implies that both the posterior estimate variance σ_t^2 and the gain parameter γ_t remain constant throughout the game and equal (see Liptser and Shiryaev (2013))

$$\gamma = \sqrt{\alpha^2 + \frac{\sigma_\mu^2}{\sigma_Y^2}} - \alpha, \text{ and } \sigma^2 = \gamma \times \sigma_Y^2. \tag{3}$$

2.2 Equilibrium

A player's (pure, public) strategy $\{a_t^i\}$ is a progressively measurable process that depends on the public information $\{Y_t\}$ and allows for public randomization. A pair of public strategies, $\{a_t^1, a_t^2\}$, is a *Perfect Public Equilibrium (PPE)* if, for each partner i at any time $\tau \geq 0$,

$$\mathbb{E}_\tau^{\{a_t^i, a_t^{-i}\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{\mu_t}{2} - c(a_t^i)\right)dt\right] \geq \mathbb{E}_\tau^{\{\widetilde{a}_t^i, a_t^{-i}\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{\mu_t}{2} - c(\widetilde{a}_t^i)\right)dt\right], \tag{4}$$

following any history, for any possible alternative strategy $\{\widetilde{a}_t^i\}$.[17]

Markov Equilibria Our model is linear: the evolution of the state μ is linear in the sum of efforts and the evolution of expected profits is linear in the state.[18] As a

[17]We discuss other notions of a strategy in Section 6. Specifically, given pure strategies, players have no private signals in the game, and any pure strategy is public. We note that a public strategy does not restrict a partner to condition only on the public expectation $\overline{\mu}_\tau$, or to revert to the equilibrium path immediately after a deviation; say, a strategy that lets a partner shirk first, and then play depending on own estimate of the fundamentals depends only on clock time and public signals. (Indeed, establishing conditions under which "double deviations" are not optimal is one of the main technical results in the paper; see Theorem 3).

[18]In Section 5 we extend the model to allow for non-linearities.

consequence, it has a unique *Markov Equilibrium*, in which play depends on the past history only via the minimal set of payoff-relevant parameters, with stationary efforts.[19]

Proposition 1 *A pair of constant strategies $\{a_t, a_t\}$ in which partners never exert effort, $a_t = 0$, for every $t \geq 0$, constitutes a PPE. It is the unique stationary PPE, and so it is the unique Markov equilibrium.*

The argument is as follows. Exploiting the linear structure of the model, we may rewrite the continuation payoffs as

$$\begin{aligned} W_\tau^i &= \mathbb{E}_\tau^{\{a_t^1,a_t^2\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{\mu_\tau}{2}e^{-\alpha(t-\tau)} + \int_\tau^t (\alpha+r)\frac{a_s^1+a_s^2}{2}e^{-\alpha(t-s)}ds - c(a_t^i)\right)dt\right] \qquad (5) \\ &= \frac{1}{2(r+\alpha)}\mathbb{E}_\tau^{\{a_t^1,a_t^2\}}[\mu_\tau] + \mathbb{E}_\tau^{\{a_t^1,a_t^2\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{a_t^1+a_t^2}{2} - c(a_t^i)\right)dt\right]. \end{aligned}$$

The first term in the last line of (5) captures the expected value of "inherited" (expected) fundamentals to a partner. Even if at some time τ partners stop exerting effort, the fundamentals will only slowly revert to zero, yielding profits all along. The second term is the forward-looking expected value of efforts undertaken in the future. Crucially, it is not affected by the fundamentals: both the marginal effect of effort on fundamentals, $(r+\alpha)\,dt$, and the marginal value of fundamentals, $\frac{1}{2(r+\alpha)}$, are constant. Thus, the effort a_M in the unique Markov equilibrium is constant, with marginal cost one half.

Our assumptions on the cost of effort normalize both the level of effort, as well as the continuation payoffs in the Markov equilibrium, to zero. We say that a partnership *unravels* if, from that point on, partners exert no more effort—that is, they play the Markov equilibrium.

We highlight that the Markov equilibrium is inefficient. As partner's effort benefits the two partners equally, the marginal social benefit of effort is twice higher than the incentives in the Markov equilibrium. In the rest of the paper we show how much of this gap can be bridged with non-Markovian, relational incentives.

[19]See Maskin and Tirole (2001) and Mailath et al. (2006) for the formal definition of Markov Equilibrium.

Strongly Symmetric Equilibria The only information about the partners' efforts is provided by the stream of profits. As both partners' efforts enter profits additively, it is not possible to identify which of the partners did, and which one did not, contribute to the common good (Fudenberg, Levine, and Maskin (1994)). Thus, as in the classic analysis of repeated duopoly by Green and Porter (1984) or of partnerships by Radner, Myerson, and Maskin (1986), it is not possible to provide incentives by "transferring" continuation value between the agents via asymmetric play, shifting resources from likely deviators.

Therefore, we concentrate throughout the paper on equilibria in which players choose symmetric strategies, conditioning the provision of effort on the public history available to them in the same way.[20] Formally, a *Strongly Symmetric Equilibrium (SSE)* is a PPE in which the strategies $\{a_t^1, a_t^2\}$ satisfy $a_\tau^1 \equiv a_\tau^2$, after every public history in $\mathcal{F}_t$.

Accounting of Incentives and Local Strongly Symmetric Equilibria Define *relational capital* w_τ as the expected discounted payoff from future efforts, that is the continuation value net of the expected value of the current fundamentals,

$$w_\tau := W_\tau - \frac{1}{2\,(r+\alpha)}\mathbb{E}_\tau^{\{a_t,a_t\}}[\mu_\tau] = \mathbb{E}_\tau^{\{a_t,a_t\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\,(a_t - c(a_t))\,dt\right]. \tag{6}$$

Relational incentives are constructed by conditioning future play, and so relational capital, on public signals. Specifically, when a partner increases effort, future high profit outcomes become more likely. For fixed strategies of the partners, this changes the probability distribution of efforts in the future. We define *relational incentive* F_τ as the marginal benefit of effort net of Markov incentives, or, equivalently, as the marginal effect of effort on relational capital. Formally,

$$F_\tau := \frac{\partial}{\partial\varepsilon}\mathbb{E}_\tau^{\{a_t,a_t\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\,(a_t - c(a_t))\,dt\right], \tag{7}$$

for revenue processes $dY_t^\varepsilon = \overline{\mu}_t^\varepsilon dt + \sigma_Y dB_t$, where $\overline{\mu}_\tau^\varepsilon = \overline{\mu}_\tau + \varepsilon\,(r+\alpha)$ and $\overline{\mu}_t^\varepsilon$ evolves as

[20] See discussion in Section 6.

in (2), with $\varepsilon > 0$.

A necessary condition for a symmetric equilibrium is that effort is locally optimal. That is, for a level of relational incentives F_τ,

$$a(F_\tau) = \arg\max_a \{(F_\tau + 1/2) \times a - c(a)\}. \tag{8}$$

A *local Strongly Symmetric Equilibrium (local SSE)* is a profile of symmetric strategies such that, following any history, actions are locally optimal, $a_\tau = a(F_\tau)$, for the function $a(\cdot)$ defined in (8), and F_τ as in (7). Finally, let $\mathcal{E}$ be the set of relational capital-incentive pairs (w_t, F_t) achievable in a local SSE. We parametrize its upper boundary by F, $F(w) = \{sup\ F_t | (w_t, F_t) \in \mathcal{E}\}$.[21]

3 Solution

This section contains the main technical results of the paper. It characterizes the set of relational incentives and relational capitals that can be delivered in a local SSE. It also constructs (nearly) optimal local SSE, and provides conditions for them to be fully incentive compatible.

As the first step, the following lemma shows how relational capital and relational incentives must evolve in a local SSE. Throughout this section, fundamentals can be determined by past efforts or stochastic, $\sigma_\mu \geq 0$, but they are imperfectly monitored, $\sigma_Y > 0$.

Lemma 1 *A symmetric strategy profile $\{a_t, a_t\}$ with bounded relational capital and relational incentives processes $\{w_t\}$ and $\{F_t\}$ is a local SSE if and only if there is a L^2 process $\{I_t\}$ such that*

$$\begin{aligned} dw_t &= (rw_t - (a_t - c(a_t)))\,dt + I_t \times (dY_t - \overline{\mu}_t dt) + dM_t^w, \\ F_\tau &= \mathbb{E}_\tau^{\{a_t,a_t\}}\left[\int_\tau^\infty e^{-(r+\alpha+\gamma)(t-\tau)}\,(r+\alpha)\,I_t dt\right], \end{aligned} \tag{9}$$

[21] F is a partial function, defined only over the relational capitals achievable in a local SSE.

and actions satisfy $a_t = a(F_t)$*, where* $\{M_t^w\}$ *is a martingale orthogonal to* $\{Y_t\}$*.*

The first equation in the lemma is a version of the "promise keeping" accounting for the continuation value (see Sannikov (2007)). If the current flow of relational capital is lower than the average promised flow, then the relational capital must deterministically increase in the next period, and vice versa. Moreover, relational capital also changes stochastically in response to the unexpected profit realizations, $dY_t - \overline{\mu}_t dt$, with linear sensitivity I_t. The martingale process captures the possibility of public randomization.

A key intuition behind the second equation in (9) is that a deviation to a higher effort today results in fundamentals above the publicly expected level not only now, but throughout the future (see, also, Prat and Jovanovic (2014), Sannikov (2014), and Prat (2015)). This means unexpectedly good news—profits higher than expected—throughout the future, which keep pushing relational capital up (when sensitivities I_t are positive).

After a deviating effort, the wedge between the private and public expectation of the fundamentals reverts to zero gradually, at a rate $\alpha + \gamma$. The first term is the exogenous rate of decay of the fundamentals. The second term is the endogenous speed of learning from profits about the fundamentals (Equation (2)). For instance, an off-equilibrium increase in effort leads to an unexpectedly high stream of profits. Upon observing it, the public attributes part of the higher profits to a permanent change in partnership's quality (due to it being stochastic) and part of it to transient luck this period (due to imperfect monitoring), used for incentives. The first effect is incorporated into higher expectation of fundamentals, and so higher expectation of profits. Hence, as the stream of higher-than-expected profits realizes, the wedge between the private and public expectation shrinks.

One way to think about the effect of learning is that, following an off-equilibrium increase in effort, the realized higher-than-expected profits are gradually attributed to a persistent exogenous change in partnership's quality (in similar fashion as in Holmström (1999)). Note also that when fundamentals are deterministic, $\sigma_\mu = 0$, partners do not learn in equilibrium, $\gamma = 0$. We discuss the effect of learning on relational incentives in more detail in Section 4.2.

Lemma 1 identifies relational capital and incentives as two variables that characterize any local SSE. The following second step establishes an HJB characterization of F, the boundary of the set of relational capital-incentive pairs achievable in a local SSE.

Theorem 1 *The upper boundary F of relational incentives achievable in a local SSE is concave and satisfies the differential equation*

$$\begin{aligned}(r+\alpha+\gamma)F(w) &= \max_I \left\{(r+\alpha)I + F'(w)\left(rw - [a(F(w)) - c(a(F(w)))]\right) + \frac{F''(w)\sigma_Y^2}{2}I^2\right\} \\ &= F'(w)\left(rw - [a(F(w)) - c(a(F(w)))]\right) - \frac{(r+\alpha)^2}{2\sigma_Y^2 F''(w)},\end{aligned} \tag{10}$$

on $[0, w^*)$, as well as the boundary conditions

$$\begin{aligned} F(0) &= 0, \\ \lim_{w\uparrow w^*} \left\{(r+\alpha+\gamma)F(w) - F'(w)\left(rw - [a(F(w)) - c(a(F(w)))]\right)\right\} &= 0, \\ \lim_{w\uparrow w} \left\{rw - [a(F(w)) - c(a(F(w)))]\right\} &= 0, \end{aligned} \tag{11}$$

where w^* is the supremum of the relational capitals achievable in a local SSE. Moreover, w^* is not attained by any local SSE.

Theorem 1 provides a characterization in the form of a differential equation and boundary conditions of the supremum F of relational incentives achievable in a local SSE. In the first line of equation (10), the left-hand side is the average flow of relational incentives needed to generate the stock of relational incentives $F(w)$, given the exponential discounting, mean reversion, and learning. The right-hand-side of (10) has the form of an HJB equation for the problem of maximizing relational incentives F as a function of relational capital. It is the point-wise maximum, over all policies I, of the expected flow of relational incentives plus the stochastic differential operator applied to F. Specifically, the first term captures the flow of relational incentives; the second term captures the change in the relational incentives resulting from the drift in relational capital; and the last term captures the loss (since the boundary is concave) resulting from the second-order variation in relational capital.

The first boundary condition in (11) says that the relational incentives in any local SSE with no relational capital must be zero—just as in the Markov equilibrium.[22] The second boundary condition is equivalent to the volatility I of the relational capital dying out close to the right boundary. It means that close to the bliss point, good outcomes are not rewarded, and bad outcomes are not punished; the flow of relational incentives dies out. The last boundary condition requires also the drift of the relational capital to die out.[23]

Theorem 1 also shows that the supremum relational capital is not attainable, and so an optimal local SSE does not exist. This follows from the last two boundary conditions in (11), which imply that the supremum relational capital would have to be the outcome of a stationary—and, hence, Markov—equilibrium. The construction of near-optimal local SSE is achieved in the following result.

Formally, the non-existence of the optimal local SSE hinges on the openness of the set of strictly positive yet arbitrarily small policies $\{I_t\}$, in the local SSE that approximate the unattainable supremum w^* (second boundary conddition in 11). To guarantee existence, we now restrict attention to a class of local SSE, with sensitivities I_t of relational capital with respect to profit flow either zero or weakly above ε, for $\varepsilon > 0$. This yields self-generation of the upper boundary F_ε in the following theorem.[24] A local SSE is called *ε-optimal* if it belongs to such class and gives rise to relational capital close to the supremum.[25]

[22]The result follows from our assumption that the Markov equilibrium effort 0 is also the lowest available effort. Allowing negative efforts and, thus, efforts below the Markov level, will allow players to "burn more value" in equilibrium and might help enlarge the set of SSE. Formally, with negative effort, the differential equation (10) and the right boundary conditions in Theorem 1 would not change, but the left boundary condition $\underline{w} \leq 0$ would become free. It is easy to establish that just as $(0, 0)$, the point $(\underline{w}, F(\underline{w}))$ must belong to the set of pairs (w, F) with zero drift.

[23]Positive drift or volatility would lead to an escape of relational capital beyond the supremum. Also, if the drift were strictly negative, one could generate relational capital above w^* simply by letting it drift down.

[24]We point out that there might be other types of approximately optimal local SSE. The class that we chose has an additional benefit of numerical tractability; in Appendix A.3 we provide the bounds on the first two derivatives of the functions F_ε in Theorem 2. In contrast, note that the equation (10) in Theorem 1 has F'' arbitrarily small close to the right boundary.

[25]Formally, we require the distance to the supremum to be vanishing in ε. The equilibria in the next theorem, in particular, achieves distance of order $O(\varepsilon^{1/3})$.

Theorem 2 *For $\varepsilon > 0$, consider local SSE with sensitivities I_t of relational capital with respect to profit flow either zero or weakly above ε. The upper boundary F_ε of relational incentives achievable in a local SSE under this constraint is concave and satisfies the differential equation*

$$(r+\alpha+\gamma)F_\varepsilon(w) = \max_{I_\varepsilon \in [\varepsilon,\infty)} \left\{ (r+\alpha)I_\varepsilon + F'_\varepsilon(w)\left(rw - [a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))]\right) + \frac{F''_\varepsilon(w)\sigma_Y^2}{2} I_\varepsilon^2 \right\} \quad (12)$$

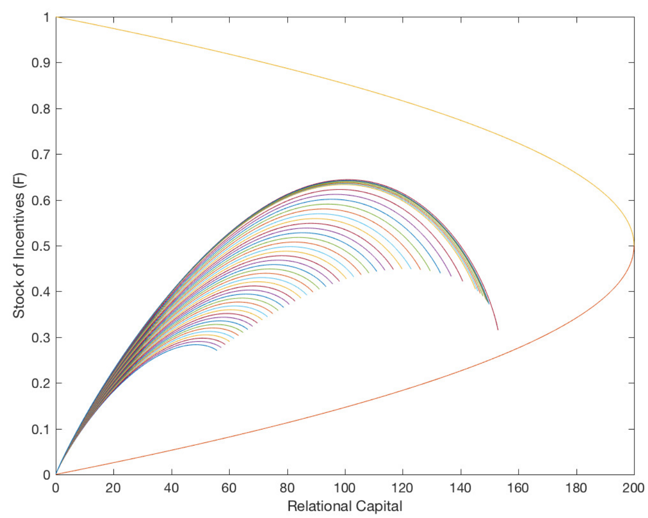
on $[0, w^*_\varepsilon]$, as well as the boundary conditions

$$\begin{aligned} F(0) &= 0, \\ (r+\alpha+\gamma)F(w^*_\varepsilon) - F'(w^*_\varepsilon)\left(rw^*_\varepsilon - [a(F(w^*_\varepsilon)) - c(a(F(w^*_\varepsilon)))]\right) &= 0, \\ rw^*_\varepsilon - [a(F(w^*_\varepsilon)) - c(a(F(w^*_\varepsilon)))] &< 0, \end{aligned} \quad (13)$$

where w^*_ε is relational capital achieved in an ε-optimal local SSE. Moreover, for any solution of differential equation (12) with boundary conditions (13) on an interval $[0, w_\varepsilon]$, there is a local SSE achieving w_ε.

The result provides a tool to find the (approximate) supremum of relational capitals achievable in local SSE. Any function solving the differential equation together with the boundary conditions defines an achievable level of relational capital. The right-most argument of the solution that reaches furthest to the right is the (approximate) supremum.

Secondly, the function F_ε in the theorem provides a recipe for constructing near-optimal local SSE (see Lemma 2 in the next section). In Figure 1, F_ε is the highest inverse parabola, which reaches furthest to the right. At any point in time, for any value of relational capital, the function determines relational incentives, and so the marginal benefit of effort. This pins down the equilibrium effort and also the relational capital in the next instant: it drifts deterministically—say, decreases if the flow benefits are large relative to the relational capital—but also responds to the stochastic news about the profit flows (see Lemma (1)). The sensitivity to those news is the one that maximizes expression (12) and, again, is pinned down by function F_ε and its second derivative. In the next instant, the game continues with updated relational capital (as described) and beliefs about the fundamentals (see Equation 2).



This figure displays many different solutions of the differential equation (12), with the near-optimal local SSE characterized by the curve that reaches farthest to the right. The horizontal parabola is the locus of the feasible relational capital-incentives pairs (w, F) that can be achieved by symmetric play in a stage game, satisfying $rw = a(F) - c(a(F))$. The efficient pair is $(200, 1/2)$.

Figure 1: Relational Incentives in a Near-optimal SSE

3.1 Verification

The full proofs of Theorems 1 and 2 are in Appendices A.2 and A.3. Here we state two crucial lemmas, which jointly correspond to the *Verification Theorem* from the stochastic control literature.[26] These lemmas highlight why we may not rely on the existing verification results.

Lemma 2 *Let* $E : [\underline{w}, \overline{w}] \to R$ *be a* C^2 *strictly concave function that satisfies the differential inequality*

[26]See, e.g., Yong and Zhou (1999), Theorem 5.5.1, for a textbook treatment.

$$(r+\alpha+\gamma)E(w) \leq \max_I \left\{ (r+\alpha)I + E'(w)\left(rw - [a(E(w)) - c(a(E(w)))]\right) + \frac{E''(w)}{2}\sigma_Y^2 I^2 \right\}$$
$$= E'(w)\left(rw - [a(E(w)) - c(a(E(w)))]\right) - \frac{(r+\alpha)^2}{2\sigma_Y^2 E''(w)}, \quad (14)$$

where a is defined in (8), together with boundary conditions for each $w^\partial \in \{\underline{w}, \overline{w}\}$:

$$E(w^\partial) \in \mathcal{E}, \quad (15)$$

or,

$$(r+\alpha+\gamma)E(w^\partial) = E'(w^\partial)\left(rw^\partial - \left[a(E(w^\partial)) - c(a(E(w^\partial)))\right]\right), \quad (16)$$
$$sgn\left(\frac{\overline{w}+\underline{w}}{2} - w^\partial\right) = sgn\left(rw^\partial - \left[a(E(w^\partial)) - c(a(E(w^\partial)))\right]\right).$$

Then each point on the curve is achieved by a local SSE, $(w_0, F(w_0)) \in \mathcal{E}$ for $w_0 \in [\underline{w}, \overline{w}]$.

The result implies that any solution E of the HJB equation (10)—equation version of (14)—with boundary conditions (15-16) provides a lower bound for the supremum F of relational incentives achievable in a local SSE. The proof also constructs local SSE, based on the maximizer in (14), that achieve relational incentives $E(w)$. Hence, the result corresponds to one half of a Verification Theorem.

In further detail, Ito's formula implies that if $\{w_t\}$ is the relational capital that follows (9) with policies $\{I_t\} = \{I^*(w_t)\}$, where $I^*(w)$ is the point-wise maximizer ("feedback control") and E is a solution to the HJB equation (10), then $\{E(w_t)\}$ is the associated process of relational incentives, as in Lemma 1. When relational capital reaches a boundary point that is known to be achievable by a local SSE, the game simply follows this local SSE. Under the alternative boundary conditions (16), relational capital is reflected back, and the construction as above continues.[27, 28]. The lemma is a convenient tool

[27] More precisely, the stock of incentives $E\left(w^\partial\right)$ at the boundary can be generated by having either $I\left(w^\partial\right) = 0$, with relational capital drifting back inside of $[\underline{w}, \overline{w}]$, or $I(w^\partial) = -2\frac{r+\alpha}{\sigma_Y^2 E''(w)} > 0$. In the proof in Appendix A.3, we show how to reduce the second case to the first one by extending the functions E and I beyond $[\underline{w}, \overline{w}]$, with $I = 0$.

[28] When (14) is an inequality, the construction is analogous, but with a continuous feedback control function $I : [\underline{w}, \overline{w}] \to \mathbb{R}_+$, $I(w) \geq I^*(w)$, for which (14) holds with equality. The existence of such function I follows from strict concavity of E.

to find non-trivial, tractable—not necessarily optimal—local SSE (see, for example, the proof of Proposition 2)

Lemma 3 *For any $\lambda > 0$, let $E^\lambda : [\underline{w}, \overline{w}] \to R$ be a concave function that satisfies the differential inequality*

$$(r+\alpha+\gamma)E^\lambda(w) \geq \max_I \left\{ (r+\alpha)I + E^{\lambda\prime}(w)\left(rw - \left[a(E^\lambda(w)) - c(a(E^\lambda(w)))\right]\right) + \frac{E^{\lambda\prime\prime}(w)\sigma_Y^2}{2} I^2 \right\} + \lambda$$
$$= E^{\lambda\prime}(w)\left(rw - \left[a(E^\lambda(w)) - c(a(E^\lambda(w)))\right]\right) - \frac{(r+\alpha)^2}{2\sigma_Y^2 E^{\lambda\prime\prime}(w)} + \lambda, \qquad (17)$$

on an interval $[\underline{w}, \overline{w}]$, together with $\left|E^{\lambda\prime}\right| \leq 1/\lambda$ and the boundary conditions $E^\lambda(\underline{w}) = F(\underline{w})$ and $E^\lambda(\overline{w}) = F(\overline{w})$. Then there is $\delta := \delta(\lambda) > 0$ such that it is not possible that the boundary F reaches locally above E^λ,

$$E^\lambda(w) < F(w) \leq E^\lambda(w) + \delta, \text{ for } w \in (\underline{w}, \overline{w}).$$

The lemma above establishes a novel escape argument and, hence, the second half of Verification Theorem for our setting. If the law of motion of w_t did not depend on the level of the value function (relational incentives), as in a standard stochastic control problem, the result would hold with $\lambda = 0$ and $\delta = \infty$. Any value above the solution of an HJB equation could be justified only by it drifting ever higher. In other words, a solution E of the HJB equation (10) with the given boundary conditions would provide an upper bound for the supremum F of relational incentives achievable in a local SSE.

In our setting, the level of the value function (relational incentives) affects the law of motion of the state variable, by determining the effort a_t. Thus, relational incentives higher than the solution to (10) may increase the right-hand side of the HJB equation. The lemma establishes only a "local" version of the bound: the boundary F of relational incentives cannot reach locally above (a local perturbation of) a solution E of the HJB equation, with given boundary conditions.

The proofs of Theorems 1 and 2 show how Lemmas 2 and 3 are sufficient to establish the characterizations. In particular, they establish that the boundaries F and F_ε are smooth and satisfy the HJB equation (10), and derive the respective boundary conditions.

3.2 Global Incentive Compatibility

So far, we have characterized local equilibria. The following result shows conditions on the primitives, under which local SSE satisfy full incentive-compatibility constraints. For simplicity, in the remaining results we assume that the cost of effort is quadratic:[29]

$$\text{(Quadratic Cost)} \qquad c(a) = \frac{1}{2}a + \frac{C}{2}a^2. \tag{18}$$

Theorem 3 *Fix $\varepsilon > 0$ and consider an $\varepsilon-$optimal local SSE $\{a_t, a_t\}$. Then, $\{a_t, a_t\}$ is an SSE when $C\sigma_Y$ is sufficiently high, where C is the second derivative of the cost function and σ_Y is observational noise.*

The problem in establishing global incentive compatibility consists in showing that, after any history, the effort choice is concave. Given that the effort cost function is strictly convex, with the second derivative C, this boils down to establishing bounds on how convex the expected benefit of effort is. Crucially, in a dynamic environment with persistence, like ours, a deviation affects the strength of incentives that the agent faces in the future. This knock-on effect makes accounting for the benefits of deviations much more involved than in a static setting, or without persistence.

Following up on this intuition, in order to bound how convex the benefit of effort is, it is sufficient to establish a uniform bound on how sensitive the relational incentives are with respect to public signals. The first part of the proof is related to the results in the literature and shows that there are no global deviations from a local SSE if this sensitivity of relational incentives is uniformly bounded (see Williams (2011), Edmans et al. (2012), Sannikov (2014), and Cisternas (2017)).

In the second part of the proof, we bound this endogenous sensitivity of relational incentives by a function of the primitives of the model. This part of the proof relies

[29] Quadratic costs greatly simplify deriving the bounds in Theorem 3 and the propositions in Section 4, but we are confident that the result can be extended to more general cost functions. Equation 41 in Appendix A.4 provides a precise sufficient condition on the parameters that guarantees global incentive compatibility.

heavily on the analytical tractability of our solution and, in particular, the bounds on derivatives F'_ε and F''_ε, established in the proof of Theorem 2. Intuitively, large noise σ_Y makes incentives costly, resulting in their low sensitivity and, thus, in their relatively linear benefit of effort.

4 Information Structure and Partnership

The informational environment of a partnership, in our setting, is determined by three parameters (see Section 2.1). First, partnerships differ by how persistent are the effects of partners' efforts and decisions. This is captured by the speed of depreciation, or mean-reversion of the fundamentals, α. Second, partnerships differ by how well their fundamentals are monitored, and so how well the progress of the venture can be tracked and assessed. This is captured by the degree of noise in the public signals, σ_Y. Third, partnerships differ by the level of uncertainty about the quality of the venture or the partners. This is captured by the degree of volatility of the fundamentals, σ_μ.

In this section, we investigate separately how the three dimensions of the information structure affect the partnership's value. (Note that efforts in the Markov equilibrium do not depend on the information structure; see Proposition 1). We begin by establishing the conditions on the informational environment for the existence of non-trivial equilibrium. At the end we extend the model to allow for career concerns and compare the effects of the informational environment on career and relational incentives.

4.1 Persistence and Non-Trivial Equilibria

Relational incentives are discounted at a rate of $r + \alpha + \gamma$, which accounts for the time preference, the persistence of the fundamentals, and learning. In the simplest case of no learning, $\gamma = 0$, and in an environment most conducing to relational incentives, when partners are patient, $r \approx 0$, the "discount rate" is then determined by the persistence of partners' effort α. The next proposition establishes that the characterization in the previous section is not vacuous, and nontrivial local SSE exist exactly when this "discount rate" is low.[30] Global incentive compatibility, when $C\sigma_Y$ is large, follows from the proof of Theorem 3.

Proposition 2 *Fix the ratio $\frac{r+\alpha+\gamma}{r}$. i) The supremum of relational capitals achievable in a SSE is strictly positive when $r+\alpha+\gamma$ is sufficiently small, and $C\sigma_Y$ is sufficiently large. ii) In contrast, if $r+\alpha+\gamma$ is sufficiently large, then the supremum of relational capitals achievable in local SSE is arbitrarily close to zero.*

The "discount rate" $r+\alpha+\gamma$ determines the time horizon for the provision of incentives (see Proposition 1). To motivate today's effort when the rate is high, high-profit outcomes must be rewarded soon—either because partners do not care much about the future, the effect of effort on profits wears off quickly, or the effect is quickly attributed to a change in partnership's quality.

Relational incentives die out when the "discount rate" is high enough for the following two reasons. The first reason is specific to relational incentives: at the bliss point of the highest relational capital, partners cannot be rewarded for high-profit outcomes. This is because rewards must be meted out in increased relational capital, which is not possible when relational capital is already at its highest. The second reason relies on the "short periods", with signals coming in continuously and hence with small precision. With poor quality of signals, instant incentives require both punishments and rewards—for bad and good signals, respectively (see Abreu, Milgrom, and Pearce (1991) and Sannikov and Skrzypacz (2007, 2010)). Intuitively, with imprecise signals, rewards and punishments are used incorrectly very often, and so both are needed for the two errors to cancel out.[31] It follows that, if the time horizon for incentive provision is short, then the partners cannot be incentivized to exert effort at the bliss point. The construction of relational incentives essentially unravels.[32]

[30] Formula 44 in Appendix A.5 provides precise sufficient conditions on the parameters that guarantee existence.

[31] In the continuous-time Gaussian setting, the incentives have even more structure: not only both rewards and punishments must be used, but relational capital must be linear in the Gaussian signal; see Proof in Appendix A.5.

[32] Relational capital at a bliss point must be low because it is a weighted average between the current benefit—close to zero when efforts are low—and the future relational capital, which can only be lower.

In contrast, with a low "discount rate" on incentives, nontrivial relational incentives are possible. In an equilibrium, good profit outcomes are always rewarded, when the relational capital of the partnership is in the workaday interior ranges. Hence, upon reaching the bliss point, partners exert effort because it will be rewarded later, once the relational capital drifts down. Waiting does not destroy much of the incentives since the discounting is low.

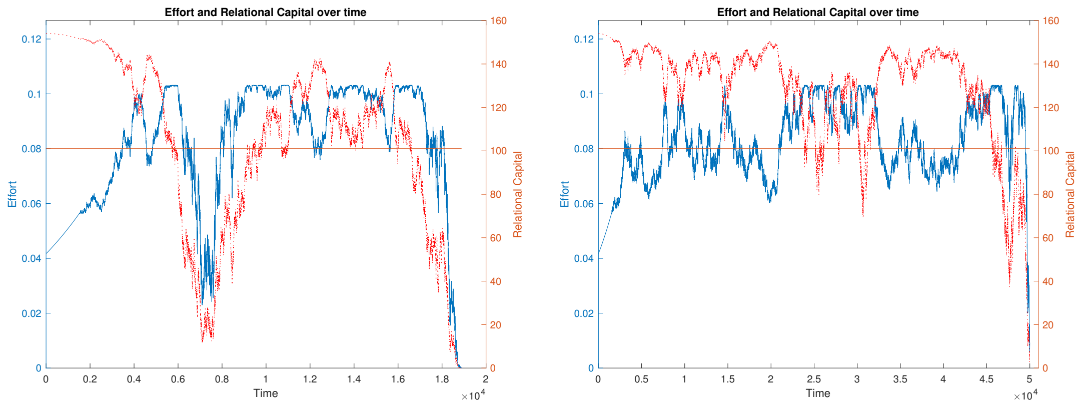
One implication is that relational incentives may increase when the effects of partners' efforts are more long-lasting and persistent (α decreases). A partnership may be a more appropriate form of organizing a firm when partners make strategic decisions that determine the future of the venture (α low), rather than make routine decisions that execute well established blueprints and keep the revenue flowing (α high).

Dynamics. This structure of incentives close to the bliss point affects the equilibrium dynamics of effort, in a non-trivial near-efficient equilibrium. Profit outcomes that exceed expectations are always good news for the partnership, increasing relational capital, as $I_t \geq 0$. However, they do not always lead to greater effort.

On one extreme, when a partnership runs out of relational capital, the partnership unravels, relational incentives disappear, and no effort is taken in the future. When relational capital is low, a good outcome that increases relational capital prolongs the life of a partnership and hence increases relational incentives and effort. Formally, function F_ϵ is increasing in this range.

On the other extreme, close to the bliss point the flow of incentives vanishes and partners are motivated by the relational incentives from the future, once relational incentives drift down. A good outcome only postpones the arrival at the interior ranges, and hence decrease relational incentives and effort. Formally, function F_ϵ is decreasing in this range.

Corollary 1 *In a near-optimal local SSE there is a threshold level of relational capital, $w^\#$, such that i) at relational capitals below $w^\#$ high profit realizations dY_t increase equilibrium effort ("rallying"), and ii) at relational capitals above $w^\#$ high profit realizations dY_t decrease equilibrium effort ("coasting").*



(a) A short-lived partnership (b) A long-lived partnership

Each panel displays a sample path of effort (on the left axis) and relational capital (on the right axis) over time, starting near the supremum relational capital. The horizontal line represents the level of relational capital at which effort is maximized. Initially players coast, and the relational capital drifts down, undisturbed by shocks. When relational capital is above the horizontal line, profit outcomes that increase relational capital lead players to exert less effort. Changes in effort and relational capital are negatively correlated. When relational capital is below the horizontal line, changes in effort and relational capital are positively correlated.

Figure 2: Effort and Relational Capital over Time

4.2 Monitoring the Partnership

The equilibrium in our dynamic environment is inefficient because partners do not observe each other's effort. Otherwise, they could sustain efficiency by reverting to the inefficient Markov equilibrium following any spat of shirking.[33] It seems thus compelling that better monitoring of the fundamentals in our game should increase the partnership's value. The following proposition shows that this intuition captures only part of the story.

Proposition 3 *i) Suppose fundamentals are deterministic, $\sigma_\mu = 0$. The supremum w^* of relational capital achievable in a local SSE is increasing in the precision of the monitoring technology σ_Y^{-1}. ii) Suppose the fundamentals are stochastic, $\sigma_\mu > 0$. The supremum w^* of relational capital achievable in a local SSE is arbitrarily close to zero, when monitoring is precise enough (σ_Y^{-1} sufficiently large). iii) Suppose the fundamentals are stochastic,*

[33]Modeling a continuous-time game with perfect monitoring runs into the usual problems. However, "Grim-Trigger" strategies approximate efficiency in a discrete-time approximation of the game, given that periods are "short" and so discount factor arbitrarily close to one, and the MinMax strategy is a stage-game Nash equilibrium.

$\sigma_\mu > 0$, and monitored perfectly, $\sigma_Y = 0$. The unique SSE is the Markov equilibrium, with relational capital $w = 0$.

When there is no uncertainty about the quality of the partnership and fundamentals are deterministic, better monitoring always improves efficiency (part (i) of the proposition). The intuition is simple: absent uncertainty about the quality, the public signals are used solely as signals of effort, and better monitoring mitigates informational frictions.[34]

When the quality of the fundamentals is uncertain, partners use public signals not only to incentivize effort but also to estimate the quality of the partnership. Better monitoring still benefits the partnership by providing better signals of efforts, but it also results in better learning about the fundamentals (higher gain parameter γ). Crucially, faster learning means that good outcomes are quickly incorporated in increased expected fundamentals, and the window for rewarding unexpectedly high outcomes, and so effort, shrinks. This shorter horizon for the incentive provision is particularly harmful at the bliss point of the partnership, when effort can be motivated solely by future rewards (Proposition 2). Part (ii) of Proposition 3 establishes that, with little noise, this negative effect is dominant and eliminates relational incentives.[35]

In the extreme, if fundamentals are monitored with no noise, the current change in fundamentals is a sufficient statistic to evaluate current effort. Incentives must be provided immediately, as in the repeated game i.i.d. setting. Since this is not possible at the bliss point of maximal relational capital, the construction of any relational incentives unravels (part (iii) of the proposition). This result is directly related to the impossibility of nontrivial incentives in Sannikov and Skrzypacz (2007, 2010).

[34]Note that providing the same level of incentives with less noise requires less variability of relational capital in equilibrium. Suppose σ_μ, $r = 1$, $\alpha = 0$; generating relational incentives F of, say, one, requires sensitivity I of relational capital to public signal equal one as well. This results in the volatility of relational capital σ_Y, increasing in noise. Formally, the only impact of σ_Y on the HJB equation (10) is through the last term, with the cost of incentives due to the second-order variation of relational capital increasing in σ_Y.

[35]Note that, as σ_Y shrinks, only the left-hand side (required mean flow of incentives) and the last term in the HJB equation 10 (contribution of the incentive flow) are scaled up. When the middle term capturing the benefit of delayed incentives disappears, the effect is similar as when the horizon for incentive provision shrinks. Formally, the solution of the HJB equation that starts around $w^* > 0$ would reach arbitrarily high levels, since i) $F''(w)$ is bounded away from negative infinity, as long as $F(w)$ is bounded away from zero; and ii) $F'(w)$ is arbitrarily steep close to w^* (Theorem 1).

One solution to this impossibility, following Abreu, Milgrom, and Pearce (1991), is to withhold the arrival of information; players observe the relevant path of signals only at times $l, 2l, 3l$, etc., for a fixed time length $l > 0$. In the new game—with "compounded" periods, actions, and signals—the horizon for incentive provision is still only the current period. However, the bundling of information improves the information quality in any given period and partners can be incentivized to work even when the relationship is at its best.

Proposition 3 highlights an alternative solution, and the benefits of *poorer* monitoring of the state. Our results show that with perfectly monitored fundamentals it is not the "short periods", but the instantaneous arrival of the relevant information and instantaneous time horizon for incentives that hampers the provision of incentives in the partnership. The impossibility in Sannikov and Skrzypacz (2007, 2010) is not due to the peculiarities of continuous-time modelling, but extreme assumption of perfect state monitoring.

4.3 Uncertainty about the Partnership

In our environment, there is an additional obstacle to partners' monitoring of each other's effort: the quality of the partnership is stochastic and unobserved by the partners. The uncertainty is captured by the degree of volatility of the fundamentals, σ_μ.[36]

Proposition 4 *The supremum of relational capital, w^*, that the partnership can generate in a local SSE decreases with the uncertainty regarding the partnership quality, σ_μ.*

In contrast to improved monitoring, reducing uncertainty about the quality of the joint venture facilitates the provision of incentives. This is because reduced uncertainty results in the public news more closely tracking the effort exerted by the partners, instead of reflecting the exogenous changes in quality.

One implication of the result is that the use of relational incentives to motivate the partners is more adequate at mature, long-standing relationships, whose quality—

[36] More precisely, variance in beliefs is strictly increasing in σ_μ, end equals zero when σ_μ does; see Equation 3.

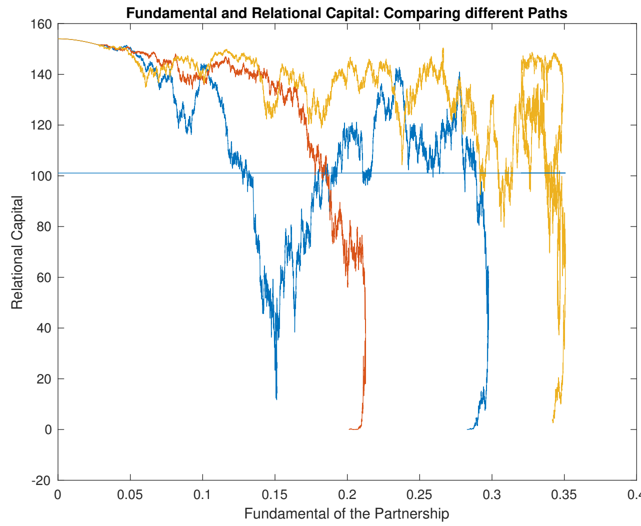
technology, product, environment, synergies, etc—is better known.[37] As the partnership is better understood, the partners can use the public news to more precisely reward the provision of effort. In contrast, in young enterprises, the uncertainty about the quality of the joint venture gets confounded with the uncertainty about the level of fundamentals. Hence, if one of the partners free rides, part of the bad news will be attributed to a "worse than expected" quality, inhibiting punishments.

Finally, depending on the level of uncertainty about the venture, a stream of bad outcomes has differential effect on relational capital and expected fundamentals. In mature partnerships with little uncertainty (σ_μ is low or absent), profit outcomes barely affect the expected fundamentals (as γ is close to zero). On the other hand, relational capital is sensitive to the public news. It follows that a profitable and mature partnership may unravel when its goodwill is tested by a short string of sharp, adverse outcome, with hardly any effect on its profitability (see Equations (2) and (3)). The Beatles (10 years together) and Daft Punk (28 years) in music; Jamie Dimon and Sandy Weill from Citigroup (15 years) in finance; and Daniel Humm and Will Guidara from Eleven Madison Park (13 years) in fine-dining provide stylized evidence. Younger enterprises, with more uncertainty about the partnership quality, tend to burn their perceived productivity before dissolution.

Corollary 2 *In a near-optimal local SSE, at any point in time t, a partnership may unravel in an arbitrarily short period of time after a sequence of unexpected bad news. The accompanying change in the expected profitability is of order σ_μ times the amount of bad news.*

Figure 3 displays the differences in the dynamics of the fundamentals and of the relational capital. It shows three different sample paths, highlighting that a partnership's relational capital is not determined by its profitability. Furthermore, even at dissolution, partnerships have different levels of the fundamentals.

[37]For tractability, in this paper, we consider only stationary models, with uncertainty constant over time. We can interpret it to be low for mature ventures, and high for the young ones.



This figure displays three different sample paths of the relational capital of a partnership, as a function of the fundamentals of the relationship. The horizontal line marks the level of relational capital at which effort is maximized.

Figure 3: Relational Capital and Fundamentals of a Partnership

4.4 Relational Incentives versus Reputation

In this paper, we have motivated the partners through relational incentives. However, going beyond our model, a key alternative incentive mechanism in weak-contractual environments is to motivate agents by reputational effects, i.e., working to build a good name for the firm (*career concerns*, Holmström (1999)).[38,39] In this section, we argue that our model can be readily extended to allow for career concerns. Importantly, the comparative statics of the effects of the informational environment on career concerns is very different from the effects on relational incentives, established above.

When building a reputation, a partner exerts effort so that good profit outcomes are misattributed to a high quality of the venture, which in turn increases the venture's

[38]Note that unlike in Holmström (1999) signal-jamming model, in our model efforts affect the state rather than the signal.

[39]An earlier version of the paper also considered a reduced form model of a spot labor market, with effort incentivized by piece-rate wages, given a monitoring/transaction cost quadratic in wages. Unlike with relational incentives, the delayed effect of effort (low α) and poor monitoring (high σ_Y) have unambiguously negative effects on the value of the venture, and high uncertainty about the partnership (high σ_μ) has a negative effect, as with relational incentives.

market value, captured by the partners. We may accommodate career concerns in our model by adjusting the cumulative profit flow of the partnership at time t, π_t, to consist of the weighted sum of the noisy signal (as before) and the public belief of the partnership fundamentals, $\pi_t = (1-\kappa)dY_t + \kappa\overline{\mu}_t$, with the parameter $\kappa \in [0,1]$ capturing the relative weight. In our baseline model, $\kappa = 0$.[40]

Compared to the direct incentives in the main model, career concerns incentives are scaled by $\frac{\gamma}{r+\alpha+\gamma}$. This is because an extra unit of effort increases the private expectation of the fundamentals above the public expectation, as in the main model. The difference in the expectations degrades over time at a rate $\alpha+\gamma$, and raises the payoff relevant parameter (public expectation) by a factor of γ, at any point in time. Finally, an increase in the payoff relevant parameter degrades over time as in the main model.[41]

It follows that, for a fixed set of parameters, the analysis of the provision of incentives in a partnership with career concerns proceeds analogously as in the main model, but with Markov incentives adjusted to $\frac{\kappa\gamma}{2(r+\alpha+\gamma)} + \frac{(1-\kappa)}{2}$. Unlike in the main model, however, Markov incentives are now affected by the information structure parameters $\alpha, \sigma_Y, \sigma_\mu$ (in line with Holmström (1999)).

Corollary 3 *With career concerns, Markov incentives are i) increasing in persistence α^{-1}, ii) increasing in the precision of the monitoring technology σ_Y^{-1}, and iii) increasing in the uncertainty about the partnership quality σ_μ.*

While persistence has a similar effect on career concerns and relational incentives, the effects of the monitoring technology and of uncertainty about the quality are in direct contrast in the two settings. First, while more precise monitoring eliminates relational incentives, it improves learning about the changes in the quality of the partnership,

[40] In Holmstrom's model, the venture can be sold in every period in a competitive market, giving the owner rights to collect the profit flow. The competitive price is the publicly expected level of the profit flows, i.e., the fundamentals, $\overline{\mu}_t$. Thus, κ may be interpreted as the fraction of the partnership traded.

[41] Combining the three effects, the total marginal benefit of effort F_{cc} due to career concerns is

$$F_{cc} = (r+\alpha) \times \frac{\gamma}{r+\alpha+\gamma} \times \frac{\kappa}{2(r+\alpha)} = \frac{\kappa\gamma}{2(r+\alpha+\gamma)}.$$

precipitating the arrival of the career concerns rewards.[42] Second, the mechanism through which the uncertainty affects the two kinds of incentives is the same, yet with opposing effects. High uncertainty crowds out the relational reward for good outcomes, which are misattributed to an exogenous change in the quality. While this restricts relational incentives, the same mechanism facilitates career-concern incentives for effort.

These contrasting comparative statics have implications on the cross-section of organizational structures. Partnerships benefit when the progress of the venture is hard to monitor, i.e. based on long-term, qualitative contributions, but the uncertainty about the venture is low. In contrast, reliance on reputation-based incentives is more fruitful if progress is easier to quantify and measure, but the uncertainty regarding the venture quality, and so the scope for building reputation, is high. This is consistent with the observation that partnerships are very common in the professional sector, i.e. law firms, accounting, and advertising (see Levin and Tadelis (2005) and Von Nordenflycht (2010)). The sector is dominated by old, established firms, with little outstanding uncertainty about them, and at the same time the product of a firm is opaque.[43]

5 Nonlinear Model

The model considered so far is linear—the evolution of fundamentals is linear in the level of fundamentals and in the exerted efforts, and the expected payoffs are linear in the level of fundamentals. While linearity helps with a tractable characterization of relational incentives in partnerships, the framework with persistence and imperfect state monitoring as well as the solution method extend to a wider class of models.

Below we consider a model with dynamics and payoffs that are nonlinear in the level of fundamentals; given the familiar difficulties with learning in nonlinear environments, we

[42] Career concerns rewards also exist at every state of the venture, with no "unraveling from the bliss point".

[43] In particular, in these knowledge-intensive environments, even after the output is produced and delivered, its quality is hard to evaluate (see Empson (2001), Greenwood and Empson (2003), Broschak (2004), and Von Nordenflycht (2010)). For instance, for an advertising agency, even after the campaign is published its quality and effects are hard to measure: Was the advertising agency's campaign responsible for the sales increase? A similar argument holds for other professional partnerships, i.e. was the lawyer's argument responsible for the acquittal?

let fundamentals be deterministic in efforts.[44] In Theorem 4, we provide the differential equation that characterizes the boundary of the local SSE incentives, analogously to Theorem 1. Then we present two canonical special cases: (i) a model of capital accumulation and (ii) oligopoly with persistence.

Suppose that the fundamentals μ_t, the publicly observable signal Y_t, and the expected profits π_i of player i follow:

$$
\begin{aligned}
d\mu_t &= g(\mu_t, a_t^1 + a_t^2)dt, \\
dY_t &= \mu_t dt + \sigma_Y dB_t^Y, \\
d\pi_t^i &= f(\mu_t, a_t^i)dt,
\end{aligned}
\tag{19}
$$

where f and g are continuously differentiable functions that are concave in the second arguments, and the discount rate is large enough to guarantee the transversality condition, $r - g_\mu > 0$. For a pair of symmetric strategies, define the continuation payoffs,

$$W_\tau = \mathbb{E}_\tau^{\{a_t,a_t\}}\left[\int_\tau^\infty e^{-r(t-\tau)} f(\mu_t, a_t), dt\right],$$

and the marginal benefit of fundamentals,

$$G_\tau := \frac{\partial}{\partial \varepsilon} W_\tau,$$

for revenue processes $dY_t^\varepsilon = \mu_t^\varepsilon dt + \sigma_Y dB_t$, where $\mu_\tau^\varepsilon = \mu_\tau + \varepsilon$ and μ_t^ε evolves as in (19), with $\varepsilon > 0$. Finally, a *local Strongly Symmetric Equilibrium* is a profile of symmetric strategies such that, following any history, actions are locally optimal,[45]

$$a_\tau := a(G_\tau, \mu_\tau) = \arg\max_a f(\mu_\tau, a) + g(\mu_\tau, a_\tau + a) \times G_\tau. \tag{20}$$

[44]Nonlinear dynamics would preclude using the relatively simple Kalman-Bucy filter to characterize learning. With learning and payoffs non-linear in fundamentals, extra effort would increase the fundamentals directly but also affect the wedge between the private and public beliefs about them. The first effect wears off only due to the mean reversion of fundamentals, while the second effect wears off also due to learning. Given different rates of discounting, proper accounting for incentives would have to include direct incentives as an additional state variable, besides the level of the fundamentals (see Section 6).

Note that in our benchmark linear model, the direct benefit of effort is constant, and so the dynamics of incentives are driven only by the wedge in beliefs. In the model of career concerns, the effort affects only the transient signal and not the hidden quality (both in the linear model, Holmström (1999), and in the nonlinear version, Cisternas (2017)), and so there is no direct effect of effort.

[45]In the linear model from Section 2.1, $g(\mu_t, a_t^1 + a_t^2) = (r+\alpha)(a_t^1 + a_t^2) - \alpha\mu_t$, $f(\mu_t, a_t^i) = \mu_t/2 - c(a_t^i)$, $w_t = W_t - \frac{\mu_t}{2(r+\alpha)}$ and $F_t = (r+\alpha)G_t - 1/2$.

Let $G(W,\mu)$ be the supremum across all local SSE of the marginal benefit of fundamentals, for a given level of continuation payoffs and fundamentals.

Theorem 4 *The upper boundary G of marginal benefit of fundamentals achievable in a local SSE is concave in W and satisfies the differential equation*

$$\begin{aligned}&(r - g_\mu(\mu, 2a))G(W,\mu) = \\ &\max_I \left\{ I + f_\mu(\mu, a) + G_W(W,\mu)\left(rW - f(\mu, a)\right) + G_\mu(W,\mu) g(\mu, 2a) + \frac{G_{WW}(W,\mu)}{2}\sigma_Y^2 I^2 \right\},\end{aligned} \tag{21}$$

at any point where G is twice continuously differentiable, with $a = a(G,\mu)$ as in (20).

Remark 1 *The result is immediately extended to settings in which the signal has drift linear in fundamentals, $dY_t = (A_t + B_t\mu_t)dt + \sigma_Y dB_t^Y$, and fundamentals are split into individual components, $\mu_t = \mu_t^1 + \mu_t^2$, with the drift of an individual component and flow expected profits equal $d\mu_t^i = g(\mu_t^i, \mu_t^{-i}, a_t^i, a_t^{-i})$ and $d\pi_t^i = f(\mu_t^i, \mu_t^{-i}, a_t^i)$.*[46]

The theorem provides an HJB characterization of the marginal benefit of fundamentals, which drives the incentives as in (20), in a nonlinear environment. As in the benchmark model, the effort in the pointwise maximization depends on the level of the maximal incentives G. The main difference, and a complication relative to Theorem 1 is that the direct marginal benefit of increased fundamentals, or Markov incentives, as well as their rate of mean reversion now depend on the level of fundamentals, which must be included as an additional state variable. This is familiar from the literature on Markov Perfect Equilibria. The additional, relational incentives are provided by the positive sensitivity I of continuation payoffs with respect to the public signals, much like in the linear environment. In particular, the marginal benefit exhibits the same persistence as in

[46] The differential equation (21) generalizes to

$$(r - g_1) \times G = \max I \left\{ I + f_1 + G_W \times (rW - f) + G_\mu \times g + \frac{G_{WW}}{2}\sigma_Y^2 I^2 \right\},$$

where functions g_1, f_1, f, g are evaluated at $(\mu/2, \mu/2, a(G,\mu), a(G,\mu))$, with

$$a(G,\mu) = \arg\max_a f(\mu/2, \mu/2, a) + g(\mu/2, \mu/2, a, a(G,\mu)) \times G.$$

the linear setting: it may exceed the Markov benefit even at a point when the sensitivity I is zero, due to the deferred relational incentives, when $G_W(W, \mu)$ is non-zero. This is a consequence of persistence and imperfect state monitoring.

The proof of the Theorem, in Appendix A.6, is based on three Lemmas, which generalize Lemmas 1-3 to nonlinear setting. Unlike Theorem 1, the result does not establish the regularity of the solution or the boundary conditions, which we believe merit a separate paper.

Capital Accumulation An important element for the provision of effort in teams, which we ignore in the main model, is that effort today may change the productivity of effort in the future. For instance, when developing a new product, early efforts to design a better product have significant effect on the productivity of later marketing efforts. Formally, the effect of effort on fundamentals may be increasing in the level of fundamentals and, hence, past efforts. Moreover, fundamentals may degrade slower (or not at all) after they ratchet above certain safe level.

Specifically, the equations in (1) generalize to:

$$d\mu_t = g(\mu_t, a_t^1 + a_t^2)dt,$$
$$dY_t = \mu_t dt + \sigma_Y dB_t^Y,$$

where g is a differentiable function that is concave in the second argument.

The differential equation (21) that characterizes the supremum of marginal benefit of fundamentals achievable in local SSE in this case is

$$(r - g_1(\mu, 2a(G, \mu)))G(W, \mu) =$$
$$\max_I \left\{ \frac{1}{2} + I + G_W\left(rW - (\mu/2 - c(a(G, \mu)))\right) + G_\mu g(\mu, 2a(G, \mu)) + \frac{G_{WW}}{2}\sigma_Y^2 I^2 \right\},$$

where the locally optimal effort $a = a(G, \mu)$ satisfies $c'(a(G, \mu)) = g_2(\mu, 2a(G, \mu)) \times G$.

Oligopoly Alternatively, we can generalize the model's payoff structure rather than changing the evolution of the fundamentals. This allows the model to speak to different economic settings, which we highlight with a simple model of an oligopoly. At any time, each firm i chooses to produce a quantity a_t^i, which adds up to the total stock of own goods produced, μ_t^i. At the same time, a fixed fraction α of total production is sold, with the mean price (inverse demand function) linearly decreasing in the quantity sold. Firms publicly observe only the price.[47] Formally, with π_t^i representing the cumulative profits of firm i until time t, we have[48]

$$\begin{aligned} d\mu_t^i &= (r+\alpha)a_t^i dt - \alpha\mu_t^i dt, \\ dY_t &= (p - \alpha(\mu_t^1 + \mu_t^2))dt + \sigma_Y dB_t^Y, \\ d\pi_t^i &= \alpha\mu_t^i dY_t - c(a_t^i)dt. \end{aligned}$$

The main difference between this model and the continuous time limit of the model analyzed by Sannikov and Skrzypacz (2007) is that here the expected price at time t, dY_t, depends on the current stock of goods, which have been produced in the past and gradually sold, rather than on the current instantaneous production (all of which is sold). The main difference with the main model in Section 2.1 of this paper is that, in a symmetric equilibrium with $a_t^1 = a_t^2, \mu_t^1 = \mu_t^2 =: \mu_t/2$, at any t, the flow revenue of a firm is $\frac{1}{2}\alpha\mu_t dY_t$, rather than $\frac{1}{2}dY_t$.

The differential equation characterizing the upper boundary of marginal benefits of fundamentals in local SSE is

$$\begin{aligned} (r+\alpha)G(W,\mu) = \max_I \Bigg\{ & -\alpha I + \alpha(p - \frac{3}{2}\alpha\mu) + \\ & G_W\left(rW - \left(\frac{1}{2}\alpha\mu(p-\alpha\mu) - c(a)\right)\right) + G_\mu(2(r+\alpha)a - \alpha\mu) + \frac{G_{WW}}{2}\sigma_Y^2 I^2\Bigg\}, \end{aligned}$$

where $a = a(G)$ is the locally optimal action that satisfies $c'(a(G)) = (r+\alpha) \times G$.

6 Concluding Remarks and Discussion

The paper characterizes optimal relational incentives in a novel model of partnership, with persistence and imperfect state monitoring. The main economic result is the benefit

[47] Each firm also observes the fraction α of own goods sold, but this does not provide any information about the competitor.

[48] For simplicity, we keep the same scaling constants as in the body of the paper.

of poorer monitoring, specific to relational incentives, which allows rewards and punishments to be delivered in a longer window of time. This insight is consistent with the predominance of well-established partnerships in industries with opaque product. It also opens up the analysis of relational incentives in a continuous-time setting. The main methodological contribution is a method to characterize optimal relational incentives that extends stochastic control, in a setting with persistence and imperfect state monitoring.

There are many ways in which one may modify our benchmark model. We conclude the paper by informally discussing the robustness of the results with respect to some of them.

1) Frequency of Play. One question is whether the results of the paper are specific to the continuous-time Brownian setting considered here, or if they hold in an approximate model with short, discrete time periods and Normal noise.

We have no reason to doubt that our results are approximated in the discrete-time setting (albeit at a cost of working with difference equations). First, the boundary of the set of incentives and relational values should be self-generating, and approximately satisfy the differential HJB characterization, with short, discrete time periods. This mirrors the classic results on Folk Theorem (see, e.g., Fudenberg, Levine, and Maskin (1994)), or the results for a Brownian Principal-Agent model (Sadzik and Stacchetti (2015)).[49, 50] Second, the problems with providing incentives close to the boundary point of the supremum relational capital persist with a high frequency of play, for information structures with Normal noise that approximate the Brownian model in this paper. This continuity result is the subject of Sannikov and Skrzypacz (2007, 2010), in the special case of no persistence (or in the case of perfectly monitored fundamentals). While we are convinced the continuity holds also in our more general setting, the formal proof is beyond the scope of this paper.

2) Mixed and Asymmetric Strategies; Other Solution Concepts. Throughout the paper

[49] In the Folk Theorem setting the boundary of the value set is approximated by a linear function; with Brownian model, the approximation requires a quadratic function.

[50] The assumption of Normality in the approximation is likely important, though; see Sadzik and Stacchetti (2015).

we restricted attention to pure strategy strongly symmetric equilibria. Formally, our representation results suggest that mixing may not happen in equilibrium, as partners face locally linear reward schemes in continuous-time, with a fixed marginal benefit of effort. With no mixing, players have no private information, and hence every strategy, even a deviating strategy, is public.[51]

In a discrete-time approximation, mixed strategies may help identify other players' strategies (see Fudenberg, Levine, and Maskin (1994)). Moreover, extending solution concept to include private communication would allow conditioning of the play on past mixing and open door to random "testing", with subsequent punishment or reward of the opponent (Kandori and Obara (2006)). For example, a partner may at random times decrease her own effort to the minimum, and let the opponent "pay" for low outputs. The same can be achieved with private, mediated messages (Rahman (2014)). This individual monitoring, in the spirit of Alchian and Demsetz (1972) and standard moral hazard, would provide additional incentives in a partnership.

Short of excluding one partner, as discussed above, asymmetric strategies neither improve the monitoring nor increase efficiency (due to convex costs). It can therefore best be thought of as an instrument for additional "burning value". While we believe asymmetric strategies may lead to better equilibria, explicitly allowing some degree of enhancing or burning of the value, discussed next, seems to us a cleaner and more direct modeling choice to capture the same channel.

3) Observable Actions. We may allow partners to exert additional observable effort. It may be productive and drive profits up, or it may be unproductive, with the only effect of "burning value". We conjecture that the only difference in the resulting differential equation 10 is the extra term in the drift of the relational capital. When observable effort is parametrized by the effect o it has on the value of the partnership, in the case when $F' < 0$, partners exert the efficient level of observable effort $\overline{o} > 0$, which drives relational capital down. When relational capital is low and $F' > 0$, partners exert the most unproductive effort $\underline{o} < 0$. This "conspicuous toiling" is best viewed as an investment in

[51] Note that a public strategy may implicitly condition on own past (pure) deviating actions, through conditioning on the public information at that point in history.

relational capital, which moves up quickly in response.[52]

4) Additional State Variables and Asymmetric Agents. Section 5 shows how the solution method may be extended to nonlinear settings, at a cost of adding fundamentals as an additional state variable. Further generalization of the result to more state variables seems straightforward. An earlier version of the paper argued for an HJB characterization of a boundary in an abstract optimization problem, with the value and a vector of state variables following

$$dV_t = [f_1(\theta_t, I_t) \times V_t + f_2(\theta_t, I_t)]dt + I_t^V dB_t^V,$$
$$d\theta_t = f_3(\theta_t, V_t, I_t)dt + \sigma(\theta_t, V_t, I_t)dB_t,$$

for some control processes $I_t^V \in \mathbb{R}, I_t \in \mathbb{R}^m$ that are progressively measurable with respect to the Brownian motions, with all functions Lipschitz continuous and f_1 positive and bounded away from zero. Importantly, the only distinction between the value and the state variables in the above system is that the value has drift linear in itself and has unrestricted volatility. Crucially, just as in the main model considered in the paper, value may affect the law of motion of the state variables.

The generalization allows for additional publicly observable variables which affect the flow payoffs or the dynamics of the fundamentals. Alternatively, it accommodates public endogenous variables, such as continuation value and incentives of some of the agents, opening up the analysis to games with asymmetric agents, persistence, and imperfectly monitored state.

[52] Similar investment in the value of a partnership has been documented in the equilibrium setting by Fujiwara-Greve and Okuno-Fujiwara (2009) and verified in the lab setting by Lee (2018).

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A Online Appendix: Proofs

A.1 Proofs of Lemmas 1, 2, and 3.

Proof of Lemma 1. The proof can be split in two parts. First, we establish that for an arbitrary pair of symmetric strategies $\{a_t, a_t\}$, relational capital $\{w_t\}$ follows a process (9), for some L^2 process $\{I_t\}$ and a martingale $\{M_t^w\}$ orthogonal to $\{Y_t\}$. The proof follows similar steps as Proposition 1 in Sannikov (2007). We derive the representation for the relational capital process in (9) in the second step.

The process $\left\{Y_t - \int_0^t \overline{\mu}_s ds\right\}$, scaled by σ_Y, is a Brownian Motion, and the process $\widetilde{w}_t = \int_0^t e^{-rs}\left(a_s - c(a_s)\right) ds + e^{-rt} w_t$ is a martingale. Since efforts, and so $\widetilde{w}_t$ are bounded, it follows from Proposition 3.4.15 in Karatzas (1991) (of which the Martingale Representation Theorem is a special case, when the filtration $\mathcal{F}_t$ is generated only by the process of profits) that $\widetilde{w}_t$ equals $\int_0^t e^{-rs} I_s \left(dY_s - \overline{\mu}_s ds\right) + M_t^w$, for an appropriate $\{I_t\}$ and a martingale $\{M_t^w\}$. Differentiating and equating both expressions for $\widetilde{w}_t$ yields the representation.

Conversely, for a bounded process $\{v_t\}$ that satisfies (9), define the process $\widetilde{v}_t = \int_0^t e^{-rs}\left(a_s - c(a_s)\right) ds + e^{-rt} v_t$, together with $\widetilde{w}_t$ as above. Both $\{\widetilde{v}_t\}$ and $\{\widetilde{w}_t\}$ are bounded martingales and so, as their values agree at infinity, they agree after every history. It follows that the processes $\{v_t\}$ and $\{w_t\}$ are the same. This establishes the first step.

Let us now evaluate the marginal benefit of effort, and the marginal relational benefit of effort F_τ in particular. Consider the Brownian Motion $\sigma_Y^{-1}\left\{Y_t - \int_0^t \overline{\mu}_s ds\right\}$. It follows from Girsanov's Theorem that the change in the underlying density measure of the output paths induced by the change in expected fundamentals from $\overline{\mu}_\tau$ to $\overline{\mu}_\tau^{dev} = \overline{\mu}_\tau + \varepsilon(r+\alpha)$ is

$$\Gamma_t^\varepsilon = e^{-\frac{1}{2}\int_\tau^t \frac{\left(\overline{\mu}_s^{dev} - \overline{\mu}_s\right)^2}{\sigma_Y^2} ds + \int_\tau^t \frac{\overline{\mu}_s^{dev} - \overline{\mu}_s}{\sigma_Y} \frac{dY_s - \overline{\mu}_s ds}{\sigma_Y}}, \tag{22}$$

for $t > \tau$, where $\{\overline{\mu}_s\}_{s\geq\tau}$ and $\{\overline{\mu}_s^{dev}\}_{s\geq\tau}$ are the associated paths of estimates, defined in (2), with $\overline{\mu}_s^{dev} - \overline{\mu}_s = \varepsilon e^{-(\alpha+\gamma)(s-\tau)}$, $s > \tau$. The relational capital at time τ thus changes to

$$\mathbb{E}_\tau^{\{a_t, a_t\}}\left[\int_\tau^\infty e^{-r(t-\tau)} \Gamma_t^\varepsilon \left(a_t - c(a_t)\right) dt\right]. \tag{23}$$

Since

$$\left.\frac{\partial}{\partial\varepsilon}\Gamma_t^\varepsilon\right|_{\varepsilon=0} = (r+\alpha)\int_\tau^t e^{-(\alpha+\gamma)(s-\tau)} \frac{dY_s - \overline{\mu}_s ds}{\sigma_Y^2},$$

it follows that

$$
\begin{aligned}
F_\tau &= \frac{\partial}{\partial \varepsilon}\mathbb{E}_\tau^{\{a_t,a_t\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\Gamma_t^\varepsilon\left(a_t - c(a_t)\right)dt\right] \\
&= (r+\alpha)\,\mathbb{E}_\tau^{\{a_t,a_t\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(a_t - c(a_t)\right)\left(\int_\tau^t e^{-(\alpha+\gamma)(s-\tau)}\frac{dY_s - \overline{\mu}_s ds}{\sigma_Y^2}\right)dt\right] \\
&= (r+\alpha)\,\mathbb{E}_\tau^{\{a_t,a_t\}}\left[\int_\tau^\infty \left(\int_t^\infty e^{-r(s-t)}\left(a_s - c(a_s)\right)ds\right)e^{-(r+\alpha+\gamma)(t-\tau)}\frac{dY_t - \overline{\mu}_t dt}{\sigma_Y^2}\right],
\end{aligned}
$$

where the last equality follows from the change of integration.

Intuitively, in the last integral above, the inside integral corresponds to the forward looking relational capital, which is then multiplied by a Brownian innovation, scaled by the discounted impact of shifted (expected) fundamentals. The correlation between the relational capital and the Brownian innovation equals I_t, from the representation of the relational capital. This yields F_τ as the expected discounted integral of I_t.

Formally, for $\tau' \geq \tau$,

$$
\begin{aligned}
\mathbb{E}_{\tau'}^{\{a_t,a_t\}}\left[F_\tau\right] &= (r+\alpha)\,\mathbb{E}_{\tau'}^{\{a_t,a_t\}}\left[\int_\tau^\infty \left(\int_t^\infty e^{-r(s-t)}\left(a_s - c(a_s)\right)ds\right)e^{-(r+\alpha+\gamma)(t-\tau)}\frac{dY_t - \overline{\mu}_t dt}{\sigma_Y^2}\right] \\
&= (r+\alpha)\left[\int_\tau^{\tau'} \left(\int_t^{\tau'} e^{-r(s-t)}\left(a_s - c(a_s)\right)ds\right)e^{-(r+\alpha+\gamma)(t-\tau)}\frac{dY_t - \overline{\mu}_t dt}{\sigma_Y^2}\right] \\
&\quad + (r+\alpha)\,w_{\tau'} \times \left[\int_\tau^{\tau'} e^{-r(\tau'-t)}e^{-(r+\alpha+\gamma)(t-\tau)}\frac{dY_t - \overline{\mu}_t dt}{\sigma_Y^2}\right] + e^{-(r+\alpha+\gamma)(\tau'-\tau)}F_{\tau'}
\end{aligned}
$$

is a martingale, as a function of τ'. Using the representation of the relational capital established above, the change of this martingale equals

$$
\begin{aligned}
&(r+\alpha)\left[e^{-(r+\alpha+\gamma)(\tau'-\tau)}I_{\tau'} + \left((a_{\tau'} - c(a_{\tau'})) - rw_{\tau'}\right)\int_\tau^{\tau'} e^{-r(\tau'-t)}e^{-(r+\alpha+\gamma)(t-\tau)}\frac{dY_t - \overline{\mu}_t dt}{\sigma_Y^2}\right] \\
&\qquad + \frac{d}{dt}e^{-(r+\alpha+\gamma)(\tau'-\tau)}F_{\tau'},
\end{aligned}
$$

where the first term is the covariance of the Brownian increments of $(r+\alpha)w_{\tau'}$ and of the bracketed stochastic intergral in the last line. Integrating over $[\tau,\infty)$ and taking expectation at time τ yields

$$
F_\tau = (r+a)\,\mathbb{E}_\tau^{\{a_t,a_t\}}\left[\int_\tau^\infty e^{-(r+\alpha+\gamma)(t-\tau)}I_t\right].
$$

Using Proposition 3.4.15 from Karatzas (1991) one more time, F_τ satisfies the above equation precisely when it can be represented as in (9).

Finally, since effort increases fundamentals by $(r+\alpha)dt$, and given the decomposition of the continuation value as in (6), the effort process is a local SSE exactly when a_t satisfies $a_t = a(F_t)$ (see e.g. the Verification Theorem in Yong and Zhou (1999) Ch.3.2). This establishes the proof.

Proof of Lemma 2. Let $I : [\underline{w}, \overline{w}] \to R$, with $I(w) \geq I^*(w) = -\frac{r+\alpha}{\sigma_Y^2 E''(w)}$, be the strictly positive continuous function that solves (14) with equality, i.e.,

$$(r+\alpha+\gamma)E(w) = (r+\alpha)I(w) + E'(w)\left(rw - [a(E(w)) - c(a(E(w)))]\right) + \frac{E''(w)}{2}\sigma_Y^2 I^2(w). \quad (24)$$

Note that the first boundary condition in (16) can hold either when $I\left(w^\partial\right) = 0$, or, as in our case, $I(w^\partial) = -2\frac{r+\alpha}{\sigma_Y^2 F''(w)} > 0$. The construction of a local SSE that achieves the boundary in the case $I(w^\partial) > 0$, when relational capital "escapes" the interval $[\underline{w}, \overline{w}]$, requires an additional step, as we detail below.

First, we extend the functions E and I beyond the boundary points w^∂, at which condition (16) is satisfied with $I\left(w^\partial\right) > 0$ as follows. Consider a boundary point $w^\partial = \overline{w}$ and, say, $rw^\partial - \left(a(E(w^\partial)) - c(a(E(w^\partial)))\right) < 0$. We use the Implicit Function Theorem to extend function E to a point $\overline{\overline{w}} > \overline{w}$, so that conditions (16) and $E''(w) < 0$ hold on $\left[\overline{w}, \overline{\overline{w}}\right]$. We also extend I continuously to the interval $\left[\overline{w}, \overline{\overline{w}}\right]$ with $I(w) = -2\frac{r+\alpha}{\sigma_Y^2 E''(w)} > 0$, so that E and I satisfy the equation (24) on $\left[\overline{w}, \overline{\overline{w}}\right]$. In words, on the interval $\left[\overline{w}, \overline{\overline{w}}\right]$ the relational incentives can be provided in two ways: they can either consist entirely of the discounted future relational incentives, with zero flow, or by providing inefficiently high flow of relational incentives I. The extension to the interval $\left[\underline{\underline{w}}, \underline{w}\right]$ in the case of $w^\partial = \underline{w}$ is analogous.

Fix $w_0 \in [\underline{w}, \overline{w}]$. We first construct a process $\{w_t\}$ of continuation values that satisfies the stochastic equation (9). Let τ^∞ be the stopping time when $\{w_t\}$ reaches a boundary point that is a local SSE. Moreover, define a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}_+}$ such that $\tau_0 = 0$; for n odd, $\tau_n \geq \tau_{n-1}$ is the stopping time when $\{w_t\}$ reaches either of the new, "outside" boundary points $\left\{\underline{\underline{w}}, \overline{\overline{w}}\right\}$; and for $n > 0$ even, $\tau_n \geq \tau_{n-1}$ is the stopping time when $\{w_t\}$ reaches either of the original "inside" boundary points $\{\underline{w}, \overline{w}\}$. For times $t \in [\tau_n, \tau_{n+1})$ with n even and $t < \tau^\infty$ we let $\{w_t\}$ be the weak solution to (9), with $I_t = I(w_t)$ and $\{M_t^w\} = 0$, starting at w_{τ_n}. Existence of a weak solution follows from the continuity of it's drift (which is a consequence of continuity of E and action defined via (8)) and volatility I (see e.g. Karatzas (1991), Theorem 5.4.22). For times $t \in [\tau_n, \tau_{n+1})$ with n odd and $t < \tau^\infty$ we let $\{w_t\}$ be the weak solution to (9), with $I_t = 0$

and $\{M_t^w\} = 0$, starting at w_{τ_n}. In words, the process $\{w_t\}$ has positive volatility until it reaches an "outside" boundary point in $\{\underline{\underline{w}}, \overline{\overline{w}}\}$, after which it drifts "inside" till it reaches the "inside" boundary point in $\{\underline{w}, \overline{w}\}$, when it resumes with the positive volatility, and so on.

It follows from Ito's formula that before τ^∞ the process $F_t = E(w_t)$, satisfies the equation in (9), with $J_t = E'(w_t) \times I(w_t)$. Since both w_t and F_t are bounded, the transversality conditions are satisfied. Finally, we may extend the processes $\{w_t\}, \{I_t\}, \{F_t\}$ and $\{J_t\}$, together with martingales $\{M_t^w\}$ beyond τ^∞ by letting them follow a local SSE that achieves $(w_{\tau^\infty}, F(w_{\tau^\infty}))$. Then the processes satisfy conditions of Lemma 1.

Proof of Lemma 3. Fix (w_0, F_0) with $w_0 \in (\underline{w}, \overline{w})$ and $E^\lambda(w_0) < F_0 < E^\lambda(w_0) + \lambda$, together with a local SSE that achieves it, and let $\{w_t\}$ and $\{F_t\}$ be the processes of relational capital and relational incentives it gives rise to. Define $D(w_t, F_t)$ as the distance of F_t from the solution E^λ of the differential equation (17),

$$D(w_t, F_t) = F_t - E^\lambda(w_t).$$

Using Ito's lemma together with Lemma 1, at any time t when $D(w_t, F_t) \in [0, \delta]$, the drift of the process $D(w_t, F_t)$ equals, for appropriate process $\{I_t\}$,

$$\begin{aligned} \frac{\mathbb{E}\left[dD(w_t, F_t)\right]}{dt} &= (r+\alpha+\gamma) F_t - (r+\alpha) I_t - E^{\lambda\prime}(w_t) \times (rw_t - (a(F_t) - c(a(F_t)))) \\ &- \frac{E^{\lambda\prime\prime}(w)\left[\sigma_Y^2 I_t^2 + d\langle M_t^w\rangle\right]}{2} \\ &\geq (r+\alpha+\gamma) F_t - (r+\alpha) I_t - E^{\lambda\prime}(w_t) \times \Big(rw_t - \Big(a(E^\lambda(w_t)) - c(a(E^\lambda(w_t)))\Big)\Big) \\ &- \frac{E^{\lambda\prime\prime}(w)\left[\sigma_Y^2 I_t^2 + d\langle M_t^w\rangle\right]}{2} - \frac{\lambda}{2} \\ &\geq (r+\alpha+\gamma)\Big(F_t - E^\lambda(w_t)\Big) + \lambda - \frac{\lambda}{2} > (r+\alpha+\gamma) \times D(w_t, F_t), \end{aligned} \tag{25}$$

The first inequality holds because $\left|E^{\lambda\prime}(w_t)\right| \leq 1/\lambda$, functions a and c are Lipschitz continuous and $D(w_t, F_t) \in [0, \delta]$, where δ is assumed to be sufficiently small. The second inequality follows because E^λ satisfies

$$\begin{aligned} (r+\alpha+\gamma)E^\lambda(w) \geq \max_I \Big\{&(r+\alpha)I + E^{\lambda\prime}(w)\Big(rw - \Big(a(E^\lambda(w)) - c(a(E^\lambda(w)))\Big)\Big) \\ &+ \frac{E^{\lambda\prime\prime}(w)\sigma_Y^2}{2} I^2\Big\} + \lambda, \end{aligned} \tag{26}$$

E^λ is concave, and $d\langle M_t^w\rangle$ is positive. Let τ be the stopping time of the process $D(w_t, F_t)$

hitting zero. Due to $D(w_0, F_0) > 0$ and inequality (25), it follows that there is a finite time T such that $E\left[D(w_T, F_T)|\tau \geq T\right] > \delta$. On the other hand, since

$$\begin{aligned} E\left[D(w_{\min\{T,\tau\}}, F_{\min\{T,\tau\}})\right] &= P\left(\tau \geq T\right) \times E\left[D(w_T, F_T)|\tau \geq T\right] \\ &+ P\left(\tau < T\right) \times E\left[D(w_\tau, F_\tau)|\tau < T\right] \\ &= P\left(\tau \geq T\right) \times E\left[D(w_T, F_T)|\tau \geq T\right], \end{aligned}$$

and the expectation is positive, it follows that $P\left(\tau \geq T\right) > 0$. This establishes that $D(w_T, F_T)$ exceeds δ with positive probability, contradiction.

A.2 Proof of Theorem 1

We begin the proof of the Theorem with the following two technical lemmas. We define the efficient level of relational capital as $w_{EF} = \frac{1}{r}\left(a_{EF} - c(a_{EF})\right)$, for the efficient effort level a_{EF}, with $c'(a_{EF}) = 1$. Let also $\underline{\mathrm{F}}$ be the lower arm of the parabola, which is the locus of the feasible relational capital-incentives pairs (w, F) that can be achieved by symmetric play in a stage game, satisfying $rw = a(\underline{F}) - c(a(\underline{F}))$; see Figure 1.

Lemma 4 *The set $\mathcal{E}$ is convex and $w^* \leq w_{EF}$. Moreover, the upper boundary F satisfies $F\left(w\right) \geq \underline{F}\left(w\right) > 0,\ w \in (0, w^*)$.*

Proof. Convexity is immediate from the possibility of public randomization, and the inequality $w \leq w_{EF}$ follows from the definitions. Finally, suppose by the way of contradiction that there exists w, $0 \leq w < w^*$, such that $F(w) < \underline{F}(w)$. Note that at w the slope of F is smaller than the slope of $\underline{F}$: otherwise, the repeated static Nash point $(0, 0)$, belonging to the graph of the convex function $\underline{F}$, and the convex set $\mathcal{E}$ would not overlap. This implies that F is bounded away below $\underline{F}$ to the right of w, and so in any local SSE the relational capital has drift bounded away above zero, as long as $w_t \geq w$ (see (9)). The possibility of escape of relational capital beyond w^* establishes the contradiction. ■

Lemma 5 *Let $E, F : [\underline{w}, \overline{w}) \to \mathbb{R}$ be two concave functions such that*

i) $F \leq E$,

ii) $F(\underline{w}) = E(\underline{w})$ *and* $F'_+(\underline{w}) = E'_+(\underline{w})$,

iii) $E''_+(\underline{w})$ *exists*

Then either $F''_+(\underline{w})$ exists and equals $E''_+(\underline{w})$ or there is G with $G(\underline{w}) = F(\underline{w})$, $G'_+(\underline{w}) = F'_+(\underline{w})$ and $G''_+(\underline{w}) < E''_+(\underline{w})$ such that $F \leq G$ in a right neighborhood of $\underline{w}$.

Proof. Suppose that $F''_+(\underline{w})$ does not exist or is not equal to $E''_+(\underline{w})$. From i), this means that there is a $\varepsilon > 0$ and a decreasing sequence $\{w_n\} \to \underline{w}$ such that

$$F(w_n) \leq E(\underline{w}) + E'_+(\underline{w}) \times (w_n - \underline{w}) + \left(E''_+(\underline{w}) - \varepsilon\right) \times (w_n - \underline{w})^2 .$$

However, concavity of F implies that the above inequality holds not only for the sequence $\{w_n\}$ but in a right neighborhood of $\underline{w}$. This implies the result, with $G(w) = E(w) - \varepsilon(w - \underline{w})^2$ in a neighborhood of $\underline{w}$. ■

The proof of Theorem 1 rests on the following four lemmas. Relying on Lemmas 2 and 3, as well as the above two lemmas, they establish that: (i) the boundary points $(w, F(w))$, for $w > 0$, may not be generated by solely deferred incentives from the future and require strictly positive volatility of relational capital, or flow of incentives; (ii) the boundary F is differentiable; (iii) given any boundary point $(w, F(w))$ and a tangent vector F', the solution of HJB equation (10) with those boundary conditions must locally lie weakly above the boundary F as well as (iv) weakly below the boundary F.

The propositions thus establish that in the range where the boundary $F(w)$ is strictly positive, it must satisfy the HJB equation (10). The proof is then concluded by establishing the boundary conditions (11).

Lemma 6 *If $(1, F')$ is a tangent vector at $(w_0, F(w_0))$, with $w_0 > 0$, then*

$$(r + \alpha + \gamma)F(w_0) > F' \times (rw_0 - [a(F(w_0)) - c(a(F(w_0)))]) . \quad (27)$$

Proof. Pick $w_0 > 0$; it follows from Lemma (4) that $F(w_0) > 0$. If the drift term is zero, $rw_0 - (a(F(w_0)) - c(a(F(w_0)))) = 0$, then (27) holds. Suppose then that the drift is strictly negative, $rw_0 - (a(F(w_0)) - c(a(F(w_0)))) < 0$ (when the inequality is reversed the proof is analogous), and such that inequality (27) fails. Assume also that $(w_0, F(w_0))$ is achieved by a local SSE, as opposed to being a limit of local SSE pairs – an assumption that we relax at the end of the proof.

Let $\overline{F}' \geq F'$ be such that (27) holds with equality, with $\overline{F}'$ in place of F'. Consider the function E defined over $[w_0, w']$, where w' is in the right neighborhood of w_0, such that E satisfies (27) with equality, with initial condition $(E(w_0), E'(w_0)) = (F(w_0), \overline{F}')$, and such that $w - (a(E(w)) - c(a(E(w)))) < 0$ for all $w \in [w_0, w']$. E is the solution of the implicit function second order ordinary differential equation.

Since $(w_0, E(w_0))$ is achieved by a local SSE and the boundary condition (11) holds at w', the function E satisfies conditions of Proposition 2, together with $I \equiv 0$. Consequently, there are local SSE that achieve every pair in its graph, and so the function lies below the boundary, $E(w) \leq F(w)$, $w \in [w_0, w']$. (Note that it follows that the inequality $(r + \alpha + \gamma)F(w_0) < F' \times (rw_0 - [a(F(w_0)) - c(a(F(w_0)))])$ is impossible, or else $\overline{F}' > F'$ and E lies above F.)

Consider now a strictly concave quadratic function G^* defined in the right neighborhood of w_0 with $(G^*(w_0), G^{*\prime}(w_0)) = (F(w_0), F'(w_0)$ and $G^*(w) < E(w)$ for $w > w_0$. The function satisfies

$$(r + \alpha + \gamma) G^*(w) < G^{*\prime}(w) (rw - [a(G^*(w)) - c(a(G^*(w)))]) - \frac{(r+\alpha)^2}{2\sigma_Y^2 G^{*\prime\prime}}, \tag{28}$$

in a right neighborhood of w_0. But then, by increasing slightly $G^{*\prime}(w_0)$, we may construct a quadratic function G over an interval $[w_0, \overline{w}]$ that also satisfies (28), together with $G(w_0) = F(w_0)$, $G'(w_0) > F'(w_0)$, and $G(\overline{w}) < F(\overline{w})$. There exists then a function $I : [w_0, \overline{w}] \to \mathbb{R}$, with $I(w) > -\frac{(r+\alpha)^2}{\sigma_Y^2 G''}$, such that

$$(r + \alpha + \gamma) G(w) = I(w) + G'(w) (rw - [a(G(w)) - c(a(G(w)))]) + \frac{G''\sigma_Y^2}{2} I^2. \ w \in [w_0, \overline{w}]$$

Applying Lemma 1, each point $(w, G(w))$, for $w \in [w_0, \overline{w}]$, can be achieved by a local SSE. Since $G'(w_0) > F'(w_0)$, this yields the desired contradiction.

Finally, when $(w_0, F(w_0))$ is not achieved by a local SSE, the result follows for the functions E, G^*, and G defined analogously as before, but with $E(w_0) = G^*(w_0) = G(w_0) = F(w_0) - \varepsilon$, for sufficiently small $\varepsilon > 0$. ■

Consider now the HJB equation (10), written as $F''(w) = \mathcal{F}(w, F, F')$. Proposition 6 implies that the right hand side of this equation is well defined and is Lipschitz continuous in the neighborhood of the points $(w_0, F(w_0), F')$, for any w_0 in $(0, w^*)$ and a tangent vector $(1, F')$, with $F'' < 0$. The following corollary is used repeatedly in the proof of the theorem:

Corollary 4 *The solution of the HJB equation (10) exists and depends continuously on the initial parameters in the neighborhood of the boundary condition* $(w_0, F(w_0), F')$*, for any* w_0 *in* $(0, w^*)$ *and a tangent vector* $(1, F')$.

Lemma 7 *The upper boundary F of the set of relational capital and relational incentives achievable in a local SSE is differentiable in* $(0, w^*)$.

Proof. Suppose to the contrary that $(w_0, F(w_0))$ is a kink. If follows from Proposition 6 that for any tangent vector $(1, F')$ at $(w_0, F(w_0))$

$$(r + \alpha + \gamma)F(w_0) > F' \times (rw_0 - [a(F(w_0)) - c(a(F(w_0)))]) .$$

Continuous dependence on the initial parameters implies that there exists $\lambda > 0$ such that $E^{\lambda *}$ solving (17) with the same initial conditions is strictly above the curve F in a neighborhood of w_0 (excluding point w_0). Invoking the continuous dependence once again, this time shifting the initial condition $(w_0, F(w_0), F')$ down to $(w_0, F(w_0) - \varepsilon, F')$, for $0 < \varepsilon << \lambda$, we construct a function E^λ that satisfies the conditions of Lemma 3, yielding a contradiction. ■

Lemma 8 *For any w_0 in $(0, w^*)$, the solution E to the differential equation (10) with initial condition $(w_0, F(w_0), F'(w_0))$ is weakly above the curve F in a neighborhood of w_0.*

Proof. Suppose to the contrary that $E < F$ in, say, the right neighborhood of w_0 (the case of the left neighborhood is analogous). From continuous dependence on the initial parameters, there are $\varepsilon, \delta > 0$ such that that the solution $\widetilde{E}$ of (10) with initial conditions $(w_0, F(w_0) - \delta, F'(w_0) + \varepsilon)$ crosses above and then comes back to F, meaning $\widetilde{E}(w_1) > F(w_1)$ and $\widetilde{E}(w_2) < F(w_2)$ for some $w_2 > w_1 > w_0$. But then the function $\widetilde{E}$ defined on $[w_0, w_2]$ satisfies conditions of Lemma 2, and so its graph is achievable by local SSE. This yields a contradiction. ■

Lemma 9 *For any w_0 in $(0, w^*)$, the solution E to the differential equation (10) with initial condition $(w_0, F(w_0), F'(w_0))$ is weakly below the curve F in a neighborhood of w_0.*

Proof. Let E satisfy (10) with initial conditions $(w_0, F(w_0), F'(w_0))$ and suppose that either $F''_+(w_0)$ does not exist, or $F''_+(w_0) \neq E''_+(w_0)$ (the case of left second derivative is analogous). Propositions 7 and 8 establish that the conditions of Lemma 5 are satisfied at w_0, and so in the right neighborhood of w_0 F is bounded above by $E(w) - \overline{\varepsilon}(w - w_0)^2$, for appropriate $\overline{\varepsilon} > 0$. Continuous dependence on initial parameters implies that there exists $\varepsilon > 0$ such that $E^{\lambda *}$ solving (17) with the same initial conditions $(w_0, F(w_0), F'(w_0))$ as E has second derivative at w_0 strictly larger than $E''(w_0) - \overline{\varepsilon}$ and is strictly above the curve F in a right neighborhood of w_0 (excluding point w_0). Invoking the continuous dependence once again, this time turning the initial condition $(w_0, F(w_0), F'(w_0))$ right to $(w_0, F(w_0), F'(w_0) - \delta)$, for $0 < \delta << \lambda$, we construct a function E^λ that satisfies the conditions of Lemma 3, yielding a contradiction. ■

The proof so far established that the boundary F satisfies the HJB equation (10) on $(0, w^*)$. To conclude the proof of the theorem, it remains to establish the boundary conditions (11).

1. $F(0) = 0$. Strictly positive relational incentives at zero in a local SSE would imply that the expected discounted efforts by each agent are strictly positive; consequently, a deviation to zero effort always would yield a nonzero relational capital to a partner, contradiction.

2. $\lim_{w \uparrow w^*} F(w) = \underline{F}(w^*)$. i) Lemma 4 shows that $\lim_{w \uparrow w^*} FD(w) < \underline{F}(w^*)$ is impossible. ii) If $\lim_{w \uparrow w^*} F(w) \in \big(\underline{F}(w_0), \overline{F}(w_0)\big)$, then, using Proposition 1, it would be possible to extend the solution to the right, with $I(w) = 0$ for $w > w^*$, contradiction. iii) If $\lim_{w \uparrow w^*} F(w) = \overline{F}(w_0)$ then, whether F approaches $\overline{F}$ from above or below, the differential equation (10) would be violated in the left neighborhood of w_*. iv) If $\lim_{w \uparrow w^*} F(w) > \overline{F}(w_0)$, then relational capital in any local SSE achieving points close to $(w^*, \lim_{w \uparrow w^*} F(w))$ has strictly positive drift, bounded away from zero. This would lead to the escape of w to the right of w^*, with positive probability.

3. $\lim_{w \uparrow w^*} F''(w) = -\infty$. When the condition is violated, then $I^*(w)$ is continuous and strictly positive close to w^*. The proof of the theorem so far establishes that F is C^2 and satisfies the differential equation (10). Given this regularity, standard verification theorem techniques establish that the equilibria achieving $(w, F(w))$, $w < w^*$, must use the optimal flow of relational incentives $I^*(w)$ a.e. (see Yong and Zhou (1999)); when $(w, F(w))$ is unattainable, the same is true for (w, E) in the limit, with E approaching $F(w)$. This, however, leads to the relational capital escaping to the right of w^*, with positive probability.

A.3 Proof of Theorem 2

In the rest of the proof, let $\overline{C}$ and $\underline{C}$ be the upper and the lower bounds on the second derivative of the cost function. In the following proofs we will need the following result.

Lemma 10 *For any $\varepsilon > 0$ and the function F_ε from Theorem 2*

$$F_\varepsilon \le \frac{(r+\alpha)^2}{256\sigma_Y^2\,(r+\alpha+\gamma)\,r^2\underline{C}^2} + 1. \tag{29}$$

Proof. Let $w_0 \in [0, \overline{w}_\varepsilon]$ be the point at which F_ε is maximized, $F_\varepsilon'(w_0) = 0$. For $w \ge w_0$ such that $F_\varepsilon(w) \ge \overline{F}(0) = 1 \ge \overline{F}(w)$, so that the drift of the relational capital $rw - (a(F_\varepsilon(w)) - c(a(F_\varepsilon(w))))$ is positive, we have

$$(r+\alpha+\gamma)\,F_\varepsilon(w) = F_\varepsilon'(w)\,(rw - [a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))]) - \frac{(r+\alpha)^2}{2\sigma_Y^2 F_\varepsilon''(w)} \le -\frac{(r+\alpha)^2}{2\sigma_Y^2 F_\varepsilon''(w)}, \tag{30}$$

$$-F_\varepsilon''(w) \le \frac{(r+\alpha)^2}{2\sigma_Y^2\,(r+\alpha+\gamma)},$$

where the equality follows from the fact that $F_\varepsilon''(w) \ge -\frac{r+\alpha}{\sigma_Y^2\varepsilon}$ (otherwse the right hand side would fall short of 1, and so the left hand side). Since $\overline{w}_\varepsilon \le w_{EF} = 1/(8rC)$, for $C \in \left[\underline{C}, \overline{C}\right]$, it therefore follows that

$$\begin{aligned} F_\varepsilon(w_0) &\le F_\varepsilon(w_0) - F_\varepsilon(\overline{w}_\varepsilon) + 1 \le \frac{1}{2}\frac{(r+\alpha)^2}{2\sigma_Y^2\,(r+\alpha+\gamma)}\left(\frac{1}{8r\underline{C}}\right)^2 + 1 \\ &= \frac{(r+\alpha)^2}{256\sigma_Y^2\,(r+\alpha+\gamma)\,r^2\underline{C}^2} + 1. \end{aligned}$$

■

The proof of the first part of the Theorem is analogous to the proof of Theorem 1. The optimal policy function implied by (12) is given by

$$\begin{aligned} I_\varepsilon^*(w) &= -\frac{r+\alpha}{\sigma_Y^2 F''(w)}, \quad \text{if } F_\varepsilon''(w) \ge -\frac{r+\alpha}{\sigma_Y^2\varepsilon} \\ I_\varepsilon^*(w) &= \varepsilon, \quad \text{if } -2\frac{r+\alpha}{\sigma_Y^2\varepsilon} < F_\varepsilon''(w) < -\frac{r+\alpha}{\sigma_Y^2\varepsilon} \\ I_\varepsilon^*(w) &= 0. \quad \text{if } F_\varepsilon''(w) \le -2\frac{r+\alpha}{\sigma_Y^2\varepsilon} \end{aligned} \tag{31}$$

It is easy to establish that $F_\varepsilon''(0) = -2\frac{r+\alpha}{\sigma_Y^2\varepsilon}$, since with any other value, the equation (12) would be violated around zero. In what follows we establish that if there is w such that $F_\varepsilon''(w) < -2\frac{r+\alpha}{\sigma_Y^2\varepsilon}$, then $F_\varepsilon'(w)$ is negative and of order $\varepsilon^{-1/3}$. We claim that this is enough to establish the proof of the Theorem. Indeed, we may define w_ε^* as the first point such that $F_\varepsilon''(w_\varepsilon^*) = -2\frac{r+\alpha}{\sigma_Y^2\varepsilon}$. Note that, crucially, the policy I_ε^* is continuous and weakly above ε over $[0, w_\varepsilon^*)$, and so with the constraint of $I \notin (0, \varepsilon)$ void. The existence of a local SSE then follows from the proof of Lemma 2. The bound follows from concavity of F_ε, and the order of $F_\varepsilon'(w_\varepsilon^*)$.

Consider w such that $F_\varepsilon''(w) < -2\frac{r+\alpha}{\sigma_Y^2\varepsilon}$. Given quadratic costs, we have

$$(a(F_\varepsilon(w)) - c(a(F_\varepsilon(w))))' = \frac{1}{C}\left(1/2 - F_\varepsilon(w)\right)F_\varepsilon'(w),$$

for $C \in \left[\underline{C}, \overline{C}\right]$. Thus, differentiating (12), we get[53]

$$\begin{aligned} F_\varepsilon''(w) &= \frac{F_\varepsilon'(w)\left(\alpha + \gamma + \left[a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))\right]'\right)}{rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))} \\ &= \frac{F_\varepsilon'(w)\left(\alpha + \gamma + \frac{1}{C}\left(1/2 - F_\varepsilon(w)\right)F_\varepsilon'(w)\right)}{rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))} \\ &\geq -\frac{F_\varepsilon'^2(w)}{2\underline{C}|rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))|}, \quad \text{when } F_\varepsilon'(w) \leq 0 \\ &\geq -\frac{C_1 F_\varepsilon'^2(w)}{\underline{C}(rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w))))}, \quad \text{when } F_\varepsilon'(w) \geq 0 \end{aligned} \tag{32}$$

where C_1 is the bound on F_ε from Lemma 10. For an appropriate $C_2 > 0$ this yields

$$\frac{F_\varepsilon'^2(w)}{|rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))|} \geq \frac{C_2}{\varepsilon}. \tag{33}$$

On the other hand, equation (12) implies that

$$F_\varepsilon'(w)\left(rw - \left[a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))\right]\right) = (r + \alpha + \gamma)\,F_\varepsilon(w) \leq (r + \alpha + \gamma)\,C_1. \tag{34}$$

Inequalities (33) and (34) imply that $|rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))| \leq C_3\varepsilon^{1/3}$, with $C_3 > 0$. Since $F_\varepsilon(w) \geq 1/2$, in the case when $F_\varepsilon'(w) \geq 0$ (so that the drift of relational capital is positive), whereas $F_\varepsilon(w) \geq \lim_{w\to w_\varepsilon^*} F_\varepsilon(w) = \underline{F}(w_\varepsilon^*) \geq C_4 > 0$, in the case when $F_\varepsilon'(w) \leq 0$ (the equality follows from the boundary condition (11)) equation (12) yields

$$|F_\varepsilon'(w)| = \frac{(r + \alpha + \gamma)\,F_\varepsilon(w)}{|rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))|} \geq C_5\varepsilon^{-1/3}. \tag{35}$$

Since F_ε is concave and bounded in $[0, C_1]$, inequality (30) implies

$$\begin{aligned} w_\varepsilon^* - w &= O(\varepsilon^{1/3}), \quad \text{when } F_\varepsilon' < 0 \\ w &= O\left(\varepsilon^{1/3}\right). \quad \text{when } F_\varepsilon' > 0 \end{aligned}$$

It is enough now to show that the case $F_\varepsilon'(w) \geq 0$ is not possible. Note that since w is small and $rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))$ positive, we have $F_\varepsilon(w) \approx \overline{F}(0) = 1$. By differentiating (32),

[53]Note that (12) implies $F_\varepsilon'(w) \times [rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))] \geq 0$.

$$
\begin{aligned}
F_\varepsilon'''(w) &= \left(\frac{F_\varepsilon'(w)}{rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))}\right)' \left(\alpha + \gamma + \left(a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))\right)'\right) \\
&+ \frac{F_\varepsilon'(w)}{rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))} \left(a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))\right)'' \\
&> \frac{F_\varepsilon'(w)}{rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))} \left(a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))\right)'' \\
&=_{sgn} \left(a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))\right)'',
\end{aligned} \tag{36}
$$

where the inequality follows from the fact that $F_\varepsilon''(w) < 0$ and

$$
\begin{aligned}
\left(rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))\right)' &= r - \frac{1}{C}\left(1/2 - F_\varepsilon(w)\right) F_\varepsilon'(w) \\
&\approx r + \frac{1}{2C} F_\varepsilon'(w) > 0, \\
\alpha + \gamma + \left(a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))\right)' &= \alpha + \gamma + \frac{1}{C}\left(1/2 - F_\varepsilon(w)\right) F_\varepsilon'(w) \\
&\approx \alpha + \gamma - \frac{1}{2C} F_\varepsilon'(w) < 0,
\end{aligned}
$$

for $C \in \left[\underline{C}, \overline{C}\right]$, when ε is small enough. Finally,

$$
\begin{aligned}
\left(a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))\right)'' &= \left(\frac{1}{C}\left(1/2 - F_\varepsilon(w)\right) F_\varepsilon'(w)\right)' \\
&=_{sgn} \left(1/2 - F_\varepsilon(w)\right) F_\varepsilon''(w) - \left(F_\varepsilon'(w)\right)^2 \\
&\approx -\frac{1}{2} F_\varepsilon''(w) - \left(F_\varepsilon'(w)\right)^2 \\
&\approx \frac{1}{4C} \frac{\left(F_\varepsilon'(w)\right)^2}{rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))} - \left(F_\varepsilon'(w)\right)^2 > 0,
\end{aligned} \tag{37}
$$

when ε is small enough, where the last line follows from (32). This establishes that $F_\varepsilon''(w^0) \leq -2\frac{r+\alpha}{\sigma_Y^2 \varepsilon}$ implies $F_\varepsilon'''(w^0) > 0$, and so the case $F_\varepsilon'(w^0) \geq 0$ is not possible. This establishes the proof of the Theorem.

Observe also that on $[0, \overline{w}_\varepsilon']$ we have the bounds

$$
\begin{aligned}
F_\varepsilon''(w) &\geq 2\frac{r+\alpha}{\sigma_Y^2 \varepsilon}, \\
F_\varepsilon'(w) \leq F_\varepsilon'(0) &\leq \frac{1}{8r\underline{C}} \times 2\frac{r+\alpha}{\sigma_Y^2 \varepsilon},
\end{aligned} \tag{38}
$$

where the second line follows from $w_{EF} \leq 1/8r\underline{C}$ and $F_\varepsilon''(w) \geq 2\frac{r+\alpha}{\sigma_Y^2 \varepsilon}$ when $F_\varepsilon'(w) \geq 0$.

A.4 Proof of Theorem 3

Step 1. Fix $\varepsilon > 0$ and consider an $\varepsilon-$optimal local SSE $\{a_t, a_t\}$, together with the processes $\{w_t\}, \{F_t\}, \{I_t\}$ and $\{J_t\}$ that satisfy equations (9) (Proposition 1 and Theorem 2). In this step we show that as long as

$$J_t \leq \frac{C\,(r + 2\,(\alpha+\gamma))}{8\,(r+\alpha)}, \quad \forall t \tag{39}$$

then, for an appropriate $X > 0$ and any deviating strategy $\{\widetilde{a}_t\}$, the relational capital at any time $\tau \geq 0$ to the deviating agent is bounded above by

$$\widetilde{w}_\tau(\widetilde{\mu}_\tau - \overline{\mu}_\tau, w_\tau) = w_\tau + \frac{F_\tau}{r+\alpha}(\widetilde{\mu}_\tau - \overline{\mu}_\tau) + X(\widetilde{\mu}_\tau - \overline{\mu}_\tau)^2. \tag{40}$$

In the formula, w_τ is the equilibrium level of relational capital, determined by (9), $\widetilde{\mu}_\tau$ are the correct beliefs, given strategies $\{\widetilde{a}_t\}$ and $\{a_t\}$, and $\overline{\mu}_\tau$ are the equilibrium beliefs, given that both strategies are $\{a_t\}$, both determined by (2). Consequently, using the bound with $\widetilde{\mu}_t = \overline{\mu}_t$, the step establishes that local SSE strategies are globally incentive compatible, as long as the bound (39) holds.

Fix a deviation strategy $\{\widetilde{a}_t\}$ and consider the process

$$v_\tau = \int_0^\tau e^{-rs}\left(\frac{\widetilde{a}_t + a_t}{2} - c(\widetilde{a}_t)\right)dt + e_\tau^{-r\tau}\widetilde{w}(\widetilde{\mu}_\tau - \overline{\mu}_\tau, w_\tau),$$

where, from (2), the wedge process $\{\widetilde{\mu}_t - \overline{\mu}_t\}$ follows

$$d\left(\widetilde{\mu}_t - \overline{\mu}_t\right) = (r+\alpha)\left(\widetilde{a}_t - a_t\right)dt - (\alpha+\gamma)\left(\widetilde{\mu}_t - \overline{\mu}_t\right)dt.$$

In order to establish that $\widetilde{w}_\tau$ bounds the relational capital under $\{\widetilde{a}_t\}$ and $\{a_t\}$, it is enough to show that the process $\{v_t\}$ has negative drift. We have

$$\begin{aligned}
e^{-rt}dv_t = {} & \left(\frac{\widetilde{a}_t + a_t}{2} - c(\widetilde{a}_t)\right)dt - r\left(w_t + \frac{F_t}{r+\alpha}(\widetilde{\mu}_t - \overline{\mu}_t) + X(\widetilde{\mu}_t - \overline{\mu}_t)^2\right) \\
& + (rW_t - (a_t + c(a_t)))dt + I_t \times (dY_t - \overline{\mu}_t dt) \\
& + \frac{\widetilde{\mu}_t - \overline{\mu}_t}{r+\alpha}\left((r+\alpha+\gamma)\,F_t - (r+\alpha)I_t dt + J_t \times (dY_t - \overline{\mu}_t dt)\right) \\
& + \left(\frac{F_t}{r+\alpha} + 2X(\widetilde{\mu}_t - \overline{\mu}_t)\right)\left((r+\alpha)\left(\widetilde{a}_t - a_t\right)dt - (\alpha+\gamma)(\widetilde{\mu}_t - \overline{\mu}_t)dt\right).
\end{aligned}$$

Given that the drift of $dY_t - \overline{\mu}_t dt$ is $(\widetilde{\mu}_t - \overline{\mu}_t)dt$, the drift of the $e^{-rt}dv_t$ process equals

$$
\begin{aligned}
&\frac{\widetilde{a}_t - a_t}{2} + c(a_t) - c(\widetilde{a}_t) + F_t(\widetilde{a}_t - a_t) \\
&+ (\widetilde{\mu}_t - \overline{\mu}_t)^2 \left(\frac{J_t}{r+\alpha} - X\left(r + 2\left(\alpha + \gamma\right)\right) \right) + (\widetilde{\mu}_t - \overline{\mu}_t)(\widetilde{a}_t - a_t) 2X\left(r+\alpha\right) \\
&\leq \frac{\widetilde{a}_t - a_t}{2} + c(a_t) - c(\widetilde{a}_t) + Ca_t(\widetilde{a}_t - a_t) \\
&+ (\widetilde{\mu}_t - \overline{\mu}_t)^2 \left(\frac{J_t}{r+\alpha} - X\left(r + 2\left(\alpha + \gamma\right)\right) \right) + (\widetilde{\mu}_t - \overline{\mu}_t)(\widetilde{a}_t - a_t) 2X\left(r+\alpha\right) \\
&= -\frac{C}{2}\left(a_t - \widetilde{a}_t\right)^2 + (\widetilde{\mu}_t - \overline{\mu}_t)^2 \left(\frac{J_t}{r+\alpha} - X\left(r + 2\left(\alpha + \gamma\right)\right) \right) \\
&+ (\widetilde{\mu}_t - \overline{\mu}_t)(\widetilde{a}_t - a_t) 2X\left(r+\alpha\right),
\end{aligned}
$$

where we used that $c(a) = \frac{1}{2}a + \frac{C}{2}a^2$, and $F_t(\widetilde{a}_t - a_t) \leq Ca_t(\widetilde{a}_t - a_t)$, with equality in the case $a_t < A$.

Note that when the matrix

$$
\begin{bmatrix} -\frac{C}{2} & X\left(r+\alpha\right) \\ X\left(r+\alpha\right) & \frac{J_t}{r+\alpha} - X\left(r + 2\left(\alpha+\gamma\right)\right) \end{bmatrix}
$$

has a positive determinant, then the trace is negative, and the matrix is negative semidefinite, guaranteing negative drift. Since

$$
\begin{aligned}
&\max_X \left\{ -\frac{C}{2} \times \left(\frac{J_t}{r+\alpha} - X\left(r + 2\left(\alpha+\gamma\right)\right) \right) - X^2\left(r+\alpha\right)^2 \right\} \\
&= \frac{C}{2\left(r+\alpha\right)} \left(\frac{C\left(r + 2\left(\alpha+\gamma\right)\right)}{8\left(r+\alpha\right)} - J_t \right),
\end{aligned}
$$

it follows that, indeed, when J_t is bounded as in (39), then $\widetilde{w}_\tau$ defined in (40) bounds the relational capital, for X that maximizes the above expression.

Step 2. Fix $\varepsilon > 0$ and consider an $\varepsilon-$optimal local SSE $\{a_t, a_t\}$. In this step we show that when $C\sigma_Y$ is sufficiently large, then for any w_t the sensitivity J_t of relational incentives is bounded as in (39). Together with step 1, this will establish the proof of Theorem 3.

Recall from Lemma 2 that

$$
J_t = J(w_t) = F_\varepsilon'(w) \times I_\varepsilon^*(w).
$$

Let us bound $I_\varepsilon^*(w)$, in the case when $F_\varepsilon'(w) > 0$. (Since $I_\varepsilon^* \geq 0$, the bound (39) holds in the case when $F_\varepsilon'(w) \leq 0$.) Over the subset $S \subseteq [0, \overline{w}_\varepsilon)$ where $F_\varepsilon''(w) < -\frac{r+\alpha}{\sigma_Y^2 \varepsilon}$, we simply have $I_\varepsilon^*(w) = \varepsilon$. Over the complement $[0, \overline{w}_\varepsilon)\backslash S$, where $F_\varepsilon''(w) \geq -\frac{r+\alpha}{\sigma_Y^2 \varepsilon}$, we have,

$$
\begin{aligned}
I_\varepsilon^*(w) &= -\frac{r+\alpha}{\sigma_Y^2 F_\varepsilon''(w)} = \frac{2}{r+\alpha}\left\{(r+\alpha+\gamma)\,F_\varepsilon(w) - F_\varepsilon'(w)\,(rw - (a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))))\right\} \\
&\leq \frac{2}{r+\alpha}\left\{\frac{(r+\alpha)^2}{256\sigma_Y^2 r^2 C^2} + r+\gamma+\alpha + \frac{r+\alpha}{4\sigma_Y^2 C r\varepsilon}\frac{1}{8C}\right\}, \\
&= \frac{r+\alpha}{128\sigma_Y^2 r^2 C^2} + \frac{2(r+\gamma+\alpha)}{r+\alpha} + \frac{1}{16\sigma_Y^2 C^2 r\varepsilon} =: I^\#
\end{aligned}
$$

where we use the bound (29) on F_ε, from Lemma 10, the bound $F_\varepsilon' \leq \frac{r+\alpha}{4\sigma_Y^2 C r\varepsilon}$ from (38), and the lower bound of $-\left(a_{EF} - c\left(a_{EF}\right)\right) = -1/8C$ on the drift of relational capital.

Condition (39) thus boils down to

$$
J_t = F_\varepsilon'(w) \times I_\varepsilon^*(w) \leq \frac{r+\alpha}{4\sigma_Y^2 C r\varepsilon} \times \left(\varepsilon + I^\#\right) \leq \frac{C\left(r+2\left(\alpha+\gamma\right)\right)}{8\left(r+\alpha\right)},
$$

or,

$$
\varepsilon + \frac{r+\alpha}{128\sigma_Y^2 r^2 C^2} + \frac{2(r+\gamma+\alpha)}{r+\alpha} + \frac{1}{16\sigma_Y^2 C^2 r\varepsilon} \leq C^2 \frac{\left(r+2\left(\alpha+\gamma\right)\right)}{2\left(r+\alpha\right)^2}\sigma_Y^2 r\varepsilon, \tag{41}
$$

which is satisfied when $C\sigma_Y$ is large enough. This concludes the proof of the step, end of the theorem.

A.5 Proofs for Section 4

Proof of Proposition 2. Part i) The proof strategy is to construct a a C^2 function $E : [0, \overline{w}] \to R$ that satisfies the differential inequality

$$
(r+\alpha+\gamma)E(w) \leq E'(w) \times (rw - [a(E(w)) - c(a(E(w)))]) - \frac{(r+\alpha)^2}{2\sigma_Y^2 E''(w)}, \tag{42}
$$

exactly as in Lemma 2, together with the left boundary condition $E\left(0\right) = 0$ (achievable by the Markov equilibrium), and the right boundary condition (16). The result then follows from Lemma 2.

Given the quadratic cost of effort $c(a) = \frac{a}{2} + \frac{C}{2}a^2$, the flow payoffs (given interior efforts) satisfy

$$
a(E) - c(a(E)) = \frac{E(w)}{2C}\left(1 - E(w)\right),
$$

and also $\underline{F}'(0) = 2Cr$. We will construct a curve E over $[0, \overline{w}]$, with $\overline{w} = \delta/r = \frac{1}{16Cr}$, constant second derivative and with the right boundary condition

$$E(\overline{w}) = \frac{1}{2} > \underline{F}(\overline{w}),$$

as well as

$$E'\left(\overline{w}\right) = \frac{(r+\alpha+\gamma)E(\overline{w})}{r\overline{w} - \frac{E(\overline{w})}{2C}\left(1-E(\overline{w})\right)} = \frac{\frac{1}{2}(r+\alpha+\gamma)}{\delta - \frac{1}{8C}} = -4C(r+\alpha+\gamma),$$

so that the first equation in (16) is satisfied at $\overline{w}$; the second equation follows from $F(\overline{w}) \in \left(\underline{F}(\overline{w}), \overline{F}(\overline{w})\right)$.

The constant second derivative D is pinned down by

$$\begin{aligned}
E\left(\overline{w}\right) &= \int_0^{\overline{w}} E'(x)dx = \int_0^{\overline{w}} \left[E'(\overline{w}) - D\left(\overline{w}-x\right)\right]dx \\
&= E'(\overline{w}) \times \frac{\delta}{r} - \frac{D}{2}\left(\frac{\delta}{r}\right)^2, \\
\frac{1}{D} &= \frac{1}{2}\frac{1}{E'(\overline{w}) \times \frac{\delta}{r} - E\left(\overline{w}\right)}\left(\frac{\delta}{r}\right)^2 = \frac{1}{2}\frac{1}{-\frac{1}{4}(r+\alpha+\gamma)\times\frac{1}{r} - \frac{1}{2}}\left(\frac{\delta}{r}\right)^2 \\
&= -\frac{2}{2+r+\alpha+\gamma}\left(\frac{\delta}{r}\right)^2.
\end{aligned}$$

It follows that, for all $w \in [0, \overline{w}]$,

$$\begin{aligned}
E(w) &\leq \frac{1}{2} + 4C(r+\alpha+\gamma) \times \frac{\delta}{r} \leq \frac{r+\alpha+\gamma}{r}, \qquad (43) \\
|E'(w)| \leq E'(0) &\leq \left|E'\left(\overline{w}\right)\right| + \frac{E\left(\overline{w}\right) - 0}{\left|E'\left(\overline{w}\right)\right|}|D| = 4C(r+\alpha+\gamma) + \frac{2+r+\alpha+\gamma}{r+\alpha+\gamma}16Cr^2, \\
rw - \frac{E(w)}{2C}&(1-E(w)) \geq -\frac{1}{8C}, \\
(r+\alpha+\gamma)E(w) - E'(w)&\left(rw - \frac{E(w)}{2C}\left(1-E(w)\right)\right) \leq \frac{(r+\alpha+\gamma)^2}{r} \\
&+ \frac{r+\alpha+\gamma}{2} + \frac{2+r+\alpha+\gamma}{r+\alpha+\gamma}2r^2, \\
-\frac{(r+\alpha)^2}{2\sigma_Y^2 D} &= \frac{(r+\alpha)^2}{2\sigma_Y^2}\frac{2}{2+r+\alpha+\gamma}\left(\frac{1}{16Cr}\right)^2 \geq \frac{2}{512\sigma_Y^2 C^2(2+r+\alpha+\gamma)},
\end{aligned}$$

where we also assume that the bound A is high enough so that the efforts are interior.

The last two inequalities in (43) establish that inequality (42) is satisfied, and so nontrivial local SSE exist, as long as

$$\frac{(r+\alpha+\gamma)^2}{r} + \frac{r+\alpha+\gamma}{2} + 2r^2\frac{2+r+\alpha+\gamma}{r+\alpha+\gamma} \leq \frac{2}{512\sigma_Y^2 C^2(2+r+\alpha+\gamma)},$$

or

$$(r+\alpha+\gamma)\frac{2+r+\alpha+\gamma}{2}\left(\frac{r+\alpha+\gamma}{r}+\frac{1}{2}+2\left(2+r+\alpha+\gamma\right)\left(\frac{r}{r+\alpha+\gamma}\right)^2\right)\leq\frac{1}{512\sigma_Y^2C^2}. \tag{44}$$

This establishes the existence of a nontrivial local SSE.

To verify the existence of fully (globally) incentive compatible nontrivial SSE, note that the policy function $I(w)$ equals zero at the extremes, and for any $w\in(0,\overline{w})$ satisfies

$$I(w)\geq-\frac{r+\alpha}{\sigma_Y^2D}\geq\frac{r+\alpha}{\sigma_Y^2}\frac{2}{2+r+\alpha+\gamma}\left(\frac{1}{16Cr}\right)^2\geq\frac{1}{256\sigma_Y^2C^2r}=:\varepsilon. \tag{45}$$

The condition (41) for global incentive compatibility, in the proof of Theorem 3, given (45), boils down to

$$\frac{1}{256\sigma_Y^2C^2r}+\frac{r+\alpha}{128\sigma_Y^2r^2C^2}+\frac{2(r+\gamma+\alpha)}{r+\alpha}+\frac{256\sigma_Y^2C^2r}{16\sigma_Y^2C^2r}\leq\frac{(r+2(\alpha+\gamma))}{512(r+\alpha)^2},$$

or

$$\frac{r+\alpha+\gamma}{r}\frac{1}{256\sigma_Y^2C^2}+\left(\frac{r+\alpha+\gamma}{r}\right)^2\frac{1}{128\sigma_Y^2C^2}+2(r+\alpha+\gamma)+16(r+\alpha)\leq\frac{1}{512}. \tag{46}$$

For a given ratio $\frac{r+\alpha+\gamma}{r}$, inequalities (44) and (46) hold when, first, $C\sigma_Y$ is sufficiently large and, second, $r+\alpha+\gamma$ is sufficiently small. This concludes the proof of the theorem.

Results do not depend of normalizing the marginal benefit of effort: The proposition remains true when the effect of action is scaled up by $X>1$, so that $d\mu_t=X(r+\alpha)(a_t^1+a_t^2)dt-\alpha\mu_t dt+\sigma_\mu dB_t^\mu$ (for example, when the effect is independent of $r+\alpha$, we have $X=(r+\alpha)^{-1}$). We briefly comment here how the proof of the proposition must be adjusted.

For a fixed $X>1$ the Markov equilibrium action becomes $a_M^X=\frac{X-1}{2C}$, and, given relational incentives F^X, the locally optimal action $a^X(F^X)$ equals $a_M^X+\frac{F^X}{C}$. The flow of relational capital (flow of equilibrium utility net of Markov equilibrium level) is $Xa^X(F^X)-c(a^X(F^X))$, which equals $\frac{F^X}{2C}\left(X-F^X\right)$; consequently, the HJB equation generalizes from (10) in Theorem 1 to

$$\begin{aligned}(r+\alpha+\gamma)F^X(w)&=\max_I\left\{X(r+\alpha)I+F^{X\prime}(w)\frac{F^X(w)}{2C}\left(X-F(w)\right)+\frac{F^{X\prime\prime}(w)\sigma_Y^2}{2}I^2\right\} \\ &=F^{X\prime}(w)\frac{F^X(w)}{2C}\left(X-F^X(w)\right)-\frac{X^2(r+\alpha)^2}{2\sigma_Y^2F^{X\prime\prime}(w)}.\end{aligned} \tag{47}$$

For the new parametrization, the construction remains analogous as in the proposition, with $a^X_{EF} = Xw^1_{EF}$, $w^X_{EF} = Xw^1_{EF}$, $\overline{w}^X = X\overline{w}^1$, $E^X(\overline{w}^X) = XE^1(\overline{w}^1)$, $E^{X\prime}(\overline{w}^X) = E^{1\prime}(\overline{w}^1)$, and $E^{X\prime\prime}(w) = \frac{1}{X}E^{1\prime\prime}(w)$. The bounds (43) in the proof change to: $\overline{E^X(w)} \leq X \times \overline{E^1(1)}$, $\overline{|E^{X\prime}(w)|} = \overline{|E^{1\prime}(w)|}$, and $\underline{(rw - (2C^{-1})E^X(w)(X - E^X(w)))} \geq X \times \underline{(rw - (2C^{-1})E^X(w)(X - E^X(w)))}$. Consequently, all the terms in the inequality (42) are bounded by the terms scaled up by X, and the inequality continues to hold.

Part ii) Fix $\underline{w} > 0$. In the proof we show that if the constant in the statement of the proposition is sufficiently high, then $w^* \leq \underline{w}$.

Suppose that $w^* > \underline{w}$. Observe that for all w such that $F'(w) \leq 0$ we have

$$F(w) \geq \lim_{s \to w^*} F(s) = \underline{F}(w^*) > \underline{F}(\underline{w}) \geq 2Cr\underline{w} =: A, \tag{48}$$

where the last inequality follows from $\underline{F}(0) = 0$, $\underline{F}'(0) = 2Cr$, and $\underline{F}$ convex. Secondly, recall from Theorem 1 that as w approaches w^* from the left, then $F'(w)$ gets arbitrarily high, and $F''(w)$ arbitrarily low. Finally, note that for any $w > 0$ the drift of the relational capital is uniformly bounded from below by

$$rw - (a(F(w)) - c(a(F(w)))) > -[a(F(w)) - c(a(F(w)))] \geq -[a_{EF} - c_{EF}] = -\frac{1}{8C} =: -B. \tag{49}$$

In the first part of the proof we establish that if the "discount factor" in the statement of the result is sufficiently high, then the the value $F(w^\#)$ of relational incentives at the point $w^\#$ such that $F'(w^\#) = 0$ would be arbitrarily high as well. We lead it to contradiction in the second part of the proof.

Fix $\overline{w}$ close to w^*, such that $-F''(\overline{w})$ equals $\varepsilon^{-1} > 0$ sufficiently large, to be determined later. Consider the differential equation

$$(r + \alpha + \gamma)A = -G'(w)B - \frac{(r+\alpha)^2}{2\sigma_Y^2 G''(w)}, \tag{50}$$

together with a boundary condition $G(\overline{w}) = F(\overline{w})$, $G''(\overline{w}) = F''(\overline{w})$, and solved for $w \leq \overline{w}$.

Let $w^{\#\#} < \overline{w}$ be such that $G'\left(w^{\#\#}\right) = 0$. We argue that

$$G'(w) > F'(w), \text{ for all } w \in [w^{\#\#}, \overline{w}]. \tag{51}$$

Indeed, note that F satisfies equation (10), related to (50), but with $F(w)$ in place of A, and $rw - (a(F(w)) - c(a(F(w))))$, in place of $-B$. It follows from (48) and (49) that (51) holds at $w = \overline{w}$. Similarly, suppose $w^\&$, $w^\# < w^\& < \overline{w}$, was the maximal point such that $G'(w^\&) \leq F'\left(w^\&\right)$. It follows that $F(w^\&) > G(w^\&)$ and $rw^\& -$

$\big(a(F(w^\&)) - c(a(F(w^\&)))\big) > -B$, and so $G''(w^\&) < F''(w^\&)$. This last inequality contradicts maximality of $w^\&$.

Crucially, inequality (51) implies that

$$F(w^\#) > G(w^{\#\#}), \tag{52}$$

for the maximal values of the respective functions, with $F'(w^\#) = 0$ and $G'(w^{\#\#}) = 0$.

We now compute $G(w^{\#\#})$. The solution to the differential equation (51) takes the form

$$G'(w) = \frac{\sqrt{\varepsilon^2 + 2c(\overline{w} - w)} - d}{c}, \quad G''(w) = -\frac{1}{\sqrt{\varepsilon^2 + 2c(\overline{w} - w)}},$$

for

$$d = 2\left(\frac{\sigma_Y}{r+\alpha}\right)^2 (r+\alpha+\gamma)\, A, \quad c = 2\left(\frac{\sigma_Y}{r+\alpha}\right)^2 B.$$

It follows that

$$\begin{aligned}
w^{\#\#} &= \overline{w} - \frac{d^2 - \varepsilon^2}{2c}, \\
G(w^{\#\#}) &= G(\overline{w}) - \int_{w^{\#\#}}^{\overline{w}} G'(w)dw = G(\overline{w}) - \int_{w^{\#\#}}^{\overline{w}} \frac{\sqrt{\varepsilon^2 + 2c(\overline{w} - w)} - d}{c} dw \\
&= G(\overline{w}) + \frac{d^2 - \varepsilon^2}{2c}\frac{d}{c} + \frac{1}{c}\frac{2}{3}\frac{1}{2c}\left[\varepsilon^2 + 2c(\overline{w} - w)\right]^{3/2}\Big|_{w^{\#\#}}^{\overline{w}} \\
&= G(\overline{w}) + \frac{d^2 - \varepsilon^2}{2c}\frac{d}{c} + \frac{\varepsilon^3}{3c^2}\frac{d^2 - \varepsilon^2}{2c} - \frac{1}{3c^2}\left(d^2 - \varepsilon^2\right)^{3/2} \geq \frac{d^3}{2c^2} - \frac{d^3}{3c^2} = \frac{1}{6}\frac{d^3}{c^2},
\end{aligned}$$

where the last inequality holds when ε is chosen small enough.

Substituting for d, c, and B in the above bound for $G(w^{\#\#})$, and using (52) we have

$$F(w^\#) \geq \frac{64}{3}\left(\frac{\sigma_Y}{r+\alpha}\right)^2 (r+\alpha+\gamma)^3 C A^3 =: D. \tag{53}$$

We now derive a contradiction from (53), when D is large enough. Let $w^\circ \in (w^\#, \overline{w})$ be such that $F(w^\circ) = \frac{1}{2}D$. Note that when, as we shall suppose,

$$\frac{1}{2}D > 1 = \overline{F}(0) \geq \overline{F}(w), \text{ for all } w \in [0, w_{EF}],$$

then for all $w \in [w^\#, w^\circ]$ the drift of relational capital $rw - (a(F(w)) - c(a(F(w))))$ is positive, and so

$$(r+\alpha+\gamma)F(w) < -\frac{(r+\alpha)^2}{2\sigma_Y^2 F''(w)}, \quad w \in [w^\#, w^\circ]$$

or

$$-F''(w) < \frac{(r+\alpha)^2}{2\sigma_Y^2(r+\alpha+\gamma)F(w)} \leq \frac{(r+\alpha)^2}{\sigma_Y^2(r+\alpha+\gamma)D}, \quad w \in [w^{\#}, w^{\circ}]. \tag{54}$$

Summarizing, when $D > 2$ we have

$$\frac{1}{2}D = F(w^{\#}) - F(w^{\circ}) < \frac{(r+\alpha)^2}{\sigma_Y^2(r+\alpha+\gamma)D}(w^{\circ} - w^{\#})^2$$
$$< \frac{(r+\alpha)^2}{\sigma_Y^2(r+\alpha+\gamma)D}\left(\frac{1}{8Cr}\right)^2,$$

where the first equality follows from the definition of w°, the first inequality follows from $F'(w^{\#}) = 0$ and the bound (54), and the lest bound follows from $w^{\circ} - w^{\#} < w_{EF} - 0 = \frac{1}{8Cr}$. Rearranging the last inequality, and substituting for D we have the necessary condition

$$1 > 32D^2C^2r^2\frac{\sigma_Y^2(r+\alpha+\gamma)}{(r+\alpha)^2} = \frac{64^3 2^6}{18}C^{10}r^8\left(\frac{\sigma_Y}{r+\alpha}\right)^6(r+\alpha+\gamma)^7\underline{w}^6, \tag{55}$$

which establishes contradiction, when $r+\alpha+\gamma$ is sufficiently large. This concludes the proof of the proposition.

Results do not depend of normalizing the marginal benefit of effort: As in the case of part i), part ii) of the proposition remains true when the effect of action is scaled up by $X < 1$, so that $d\mu_t = X(r+\alpha)(a_t^1 + a_t^2)dt - \alpha\mu_t dt + \sigma_\mu dB_t^\mu$ (for example, when the effect is independent of $r+\alpha$, we have $X = (r+\alpha)^{-1}$). We briefly comment here how the proof of the proposition must be adjusted.

Fix $X < 1$; the bounds in the proposition change to $A^X = \frac{1}{X}A^1$, $B^X = X \times B^1$, and the last term in the equation (50) is scaled up by X^2 (see (47)). Consequently, $d^X = \frac{1}{X^3}d^1$, $c^X = \frac{1}{X} \times c^1$, $G^X(w^{X\#\#}) = \frac{1}{X^7}G^1(w^{1\#\#})$. This results in in bounds $-\overline{F^{X\prime\prime}(w)} \leq -\overline{F^{1\prime\prime}(w)} \times X^9$ and, rearranging terms, the right-hand side in the necessary inequality (55) is multiplied by $X^{18} < 1$.

Proof of Proposition 3. Part i) Suppose $\gamma = \sigma_\mu = 0$. We show that the supremum w_ε^* of relational capitals achievable in the $\varepsilon-$optimal local SSE is increasing in σ_Y^{-1}, for every $\varepsilon > 0$. Note that decreasing σ_Y changes equation (14) in Proposition 2 only by decreasing the last term. This means that if a pair of functions (F, I) satisfies the conditions of Lemma 2 for some interval $[\underline{w}, \overline{w}]$ and a given σ_Y, then for any σ_Y' with $0 < \sigma_Y' < \sigma_Y$ there is a function $\overline{I} \geq I$ such that the pair $\left(F, \overline{I}\right)$ satisfies the conditions of Lemma 2 for σ_Y'. Applying the result to the pair $(F_\varepsilon, I_\varepsilon^*)$ on the interval $[0, \overline{w}_\varepsilon]$ as in the proof of Theorem 2, for any $\varepsilon > 0$, establishes the proof.

Part ii) Fix $\underline{w} > 0$. Proof of part ii) of Proposition 2 establishes that a necessary condition for $w^* \geq \underline{w}$ is inequality (55), reproduced below:

$$1 > 32D^2C^2r^2\frac{\sigma_Y^2(r+\alpha+\gamma)}{(r+\alpha)^2} = \frac{64^3 2^6}{18}C^{10}r^8\left(\frac{\sigma_Y}{r+\alpha}\right)^6(r+\alpha+\gamma)^7\underline{w}^6. \tag{56}$$

Recall also that, when σ_Y is close to zero, γ is of order σ_Y^{-1} (see equation (3)). Substituting, the right hand side of (55) is of order σ_Y^{-1}, when σ_Y is close to zero. This establishes that $w^* \leq \underline{w}$, when σ_Y is sufficiently small.

Part iii). As a preliminary step, we show that a symmetric strategy profile $\{a_t, a_t\}$ is an SSE with associated relational capital process $\{w_t\}$ if and only if there is an L^2 process $\{I_t\}$ such that

$$dw_t = (rw_t - (a_t - c(a_t)))\,dt + I_t \times (d\mu_t - [(r+\alpha)\,2a_t - \alpha\mu_t]\,dt) + dM_t^w, \tag{57}$$

where $a_t = a((r+\alpha)\,I_t)$, and $\{M_t^w\}$ is a martingale orthogonal to $\{Y_t\}$, and the transversality condition $\mathbb{E}\left[e^{-rt}w_t\right] \to_{t\to\infty} 0$ holds.

The proof is identical to the first part of the proof of Lemma 1: since the process $\left\{\mu_t - \int_0^t [(r+\alpha)\,2a_s - \alpha\mu_s]\,ds\right\}$, scaled by σ_μ, is a Brownian Motion, it follows from Proposition 3.4.15 in Karatzas (1991) that a process $\{w_t\}$ is the relational capital process associated with $\{a_t, a_t\}$, defined in (6), precisely when it can be represented as in (57), for some L^2 process $\{I_t\}$ and a martingale $\{M_t^w\}$ orthogonal to $\{\mu_t\}$.

As regards incentive compatibility, fix an alternative strategy $\{\widetilde{a}_t\}$ for player i and note that the relational capital satisfies

$$\begin{aligned}
&\mathbb{E}_\tau^{\{\widetilde{a}_t,a_t\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{\widetilde{a}_t+a_t}{2} - c(\widetilde{a}_t)\right)dt\right] \\
&= \mathbb{E}_\tau^{\{\widetilde{a}_t,a_t\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{\widetilde{a}_t+a_t}{2} - c(\widetilde{a}_t)\right)dt + w_\tau + \int_\tau^\infty d\left(e^{-rt}w_t\right)\right] \\
&= w_\tau + \mathbb{E}_\tau^{\{\widetilde{a}_t,a_t\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{\widetilde{a}_t+a_t}{2} - c(\widetilde{a}_t)\right)dt + \int_\tau^\infty e^{-rt}\left(dw_t - rw_t dt\right)\right] \\
&= w_\tau + \mathbb{E}_\tau^{\{\widetilde{a}_t,a_t\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{\widetilde{a}_t-a_t}{2} - c(\widetilde{a}_t) + c\left(a_t\right) + I_t\left(r+\alpha\right)\left(\widetilde{a}_t - a_t\right)\right)dt\right],
\end{aligned}$$

where the first equality follows from $\mathbb{E}_\tau^{\{\widetilde{a}_t^i,a_t^{-i}\}}\left[e^{-r(t-\tau)}w_t\right] \to 0$, as $t \to \infty$ (given that efforts are bounded), and the last one follows from $\mathbb{E}_\tau^{\{\widetilde{a}_t,a_t\}}\left[d\mu_t - [(r+\alpha)\,2a_t - \alpha\mu_t]\,dt\right] =$

$(r+\alpha)\,\mathbb{E}_{\tau}^{\{\widetilde{a}_t,a_t\}}\left[\widetilde{a}_t-a_t\right]$. Since continuation value and relational capital differ by a constant, it follows from this representation and convexity of costs that there exists no profitable deviating strategy for partner i if and only if her effort process satisfies $a_t = a((r+\alpha)\,I_t)$.

We are now ready to establish part iii) of the proposition. From representation (57) it follows that when $w_t \geq \varepsilon > 0$, then either the volatility satisfies $I_t\sigma_\mu \geq \delta > 0$, in order to incentivize a strictly positive, more efficient effort, or the drift satisfies $\mathbb{E}_{\tau}^{\left\{a_t^1,a_t^2\right\}}\left[dw_t\right] \geq \delta > 0$, to satisfy promise keeping (where δ depends on ε). It follows that if $w_0 > 0$ then the process $\{w_t\}$ escapes to infinity with positive probability, which, given bounded efforts, yields contradiction.

Proof of Proposition 4. Fix $\sigma_\mu > \sigma_\mu^{\#} \geq 0$; we show that, for any $\varepsilon > 0$, the corresponding suprema of relational capitals achievable in the $\varepsilon-$optimal local SSE satisfy $w_\varepsilon^{\#*} \geq w_\varepsilon^*$. The proof is very related to the proof of Proposition 3. One extra complication is that now, changing the noise of the fundamentals also affects the boundary conditions (16) in Lemma 2, via the effect on γ.

Specifically, note that decreasing σ_μ changes equation (14) in Proposition 2 only by decreasing γ in the first term. Let $\gamma^{\#} \leq \gamma$ be the two corresponding gain parameters, and let $\overline{w}_\varepsilon$ be the relational capital achievable in a $\varepsilon-$optimal local SSE with σ_μ, as in the proof of Theorem 2, together with a pair of functions $(F_\varepsilon, I_\varepsilon^*)$ defined on $[0,\overline{w}_\varepsilon]$. Let $\left(F_\varepsilon^{\#}, I_\varepsilon^{\#*}\right)$ extend the functions $(F_\varepsilon, I_\varepsilon^*)$ to the right by letting $F_\varepsilon^{\#''}(w) = F_\varepsilon^{\#''}(\overline{w}_\varepsilon)$ and $I_\varepsilon^{\#*}(w) = I_\varepsilon^{\#*}(\overline{w}_\varepsilon)$, for $w > \overline{w}_\varepsilon$, and let $\overline{w}_\varepsilon^{\#}$ be the first argument such that the boundary condition (16) is satisfied. The existence of such $\overline{w}_\varepsilon^{\#}$ follows from the fact that at $\overline{w}_\varepsilon$ condition (16) is violated, with the left-hand-side too small (due to $\gamma^{\#} \leq \gamma$), and when w increases and $F_\varepsilon^{\#}$ decreases and approaches from above the value $\underline{F}(w)$ at which the drift dies out, the left-hand-side is bounded away from zero, and the right-hand-side converges to zero. It follows that for small $\sigma_\mu^{\#}$ and $\gamma^{\#}$, the pair $\left(F_\varepsilon^{\#}, I_\varepsilon^{\#*}\right)$ satisfies conditions of Lemma 2. This establishes the proof.

A.6 Proof of Theorem 4

The concavity of G in continuation value follows, as usual, from the availability of public randomization. Below, first, in Lemma 11 we derive the continuation value and marginal benefit of fundamentals processes in local SSE, analogously as in Lemma 1. Then, Lemma

12 establishes that at any point where G is sufficiently smooth, the inequality version of (21) must hold, with the left hand side greater than the right hand side. Otherwise, intuitively, function G can be locally increased and still satisfy the inequality, despite the dependence of the right-hand-side on the value G; the function provides a recipe to construct a local SSE process, using the Ito formula (compare Lemma 2). Conversely, Lemma 13 establishes the reverse weak inequality, using an "escape argument" similar as in Lemma 3. Lemmas 12 and 13 hence establish the proof of the Theorem.

Lemma 11 *A symmetric strategy profile* $\{a_t, a_t\}$ *with bounded processes of continuation value and marginal benefit of fundamentals processes* $\{W_t\}$ *and* $\{G_t\}$ *is a local SSE if and only if there are* L^2 *processes* $\{I_t\}$, $\{J_t\}$ *such that*

$$\begin{aligned} dW_t &= (rW_t - f(\mu_t, a_t))\, dt + I_t \times (dY_t - \mu_t dt) + dM_t^W, \\ dG_t &= (r - g_\mu(\mu_t, 2a_t))\, G_t dt - (I_t + f_\mu(\mu, a_t))dt + J_t \times (dY_t - \mu_t dt) + dM_t^G, \end{aligned} \tag{58}$$

and actions satisfy $a_t = a(G_t)$*, where* $\{M_t^W\}$ *and* $\{M_t^G\}$ *are martingales orthogonal to* $\{Y_t\}$*.*

Proof. The proof for the process of continuation values is entirely analogous to the first part of the proof of Lemma 1, based on Proposition 3.4.14 in Karatzas (1991), and hence is omitted here. The proof for the process of marginal benefit of fundamentals is analogous too, with the following changes.

First, the generic formula (22) for the relative density under fundamentals shifted from μ_τ to $\mu_\tau + \varepsilon$ remains the same,

$$\Gamma_t^\varepsilon = e^{-\frac{1}{2}\int_\tau^t \frac{\left(\mu_s^{dev} - \mu_s\right)^2}{\sigma_Y^2} ds + \int_\tau^t \frac{\mu_s^{dev} - \mu_s}{\sigma_Y} \frac{dY_s - \mu_s ds}{\sigma_Y}},$$

for $t > \tau$, where $\{\mu_s\}_{s \geq \tau}$ and $\{\mu_s^{dev}\}_{s \geq \tau}$ are the associated paths of relational capital that satisfy (19), with $\mu_s^{dev} - \mu_s \approx \varepsilon e^{\int_\tau^s g_\mu(\mu_v, 2a_v)dv}$, for small ε and $s > \tau$. The continuation value at time τ thus changes to

$$\mathbb{E}_\tau^{\{a_t, a_t\}} \left[\int_\tau^\infty e^{-r(t-\tau)} \Gamma_t^\varepsilon f(\mu_t^{dev}, a_t) dt \right],$$

and the marginal benefit of fundamentals satisfies

$$G_\tau = \frac{\partial}{\partial \varepsilon}\mathbb{E}_\tau^{\{a_t,a_t\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\Gamma_t^\varepsilon f(\mu_t^{dev}, a_t)dt\right] \tag{59}$$

$$= \mathbb{E}_\tau^{\{a_t,a_t\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(e^{\int_\tau^t g_\mu(\mu_s,2a_s)ds} f_\mu(\mu_t,a_t) + f(\mu_t,a_t)\int_\tau^t e^{\int_\tau^s g_\mu(\mu_v,2a_v)dv}\frac{dY_s-\mu_s ds}{\sigma_Y^2}\right)dt\right]$$

$$= \mathbb{E}_\tau^{\{a_t,a_t\}}\left[\int_\tau^\infty e^{\int_\tau^t (g_\mu(\mu_s,2a_s)-r)ds}\left(f_\mu(\mu_t,a_t)dt + \left(\int_t^\infty e^{-r(s-t)}f(\mu_s,a_s)ds\right)\frac{dY_t-\mu_t dt}{\sigma_Y^2}\right)\right].$$

Proceeding as in the proof of Lemma 1, with $f(\mu_s, a_s)$ replacing $a_s - c(a_s)$ and $\int_\tau^t (g_\mu(\mu_s, 2a_s) - r)ds$ replacing $-(r+\alpha+\gamma)(t-\tau)$, we establish that

$$G_\tau - \mathbb{E}_\tau^{\{a_t,a_t\}}\left[\int_\tau^\infty e^{\int_\tau^t (g_\mu(\mu_s,2a_s)-r)ds} f_\mu(\mu_t,a_t)dt\right] = \mathbb{E}_\tau^{\{a_t,a_t\}}\left[\int_\tau^\infty e^{\int_\tau^t (g_\mu(\mu_s,2a_s)-r)ds} I_t\right].$$

Using Proposition 3.4.14 from Karatzas (1991) one more time, the marginal benefit of fundamentals satisfies the above equation precisely when it can be represented as in 58. ■

Lemma 12 *At any point where G is twice continuously differentiable, the left-hand-side of equation (21) is weakly greater than the right-hand-side.*

Proof. Suppose instead that there is a point (W^*, μ^*) at which G is twice continuously differentiable and the left-hand-side of (21) is strictly lower than the right-hand side. From twice continuous differentiability of G and given that the functions f_μ, f, and g are continuous, there exists a smooth function h defined over a neighborhood S of (W^*, μ^*) such that i) on S the left-hand-side of (21) is strictly lower than the right-hand side, when applied to $G + h$, ii) $h(W^*, \mu^*) > 0$, iii) h is strictly concave in W, and iv) $h = 0$ on ∂S. Let $\overline{G} := G + h$.

Let $(W_0, \mu_0) = (W^*, \mu^*)$; we first construct the process $\{W_t, \mu_t\}$ that satisfies the first equations in (19) and (58) as follows. Let τ be the stopping time when $\{W_t, \mu_t\}$ hits ∂S. For any $(W, \mu) \in S$ let function $I(W, \mu)$ be such that

$$(r - g_\mu(\mu, 2a(\overline{G}))) \times \overline{G}(W,\mu) = \tag{60}$$

$$I(W,\mu) + f_\mu(\mu, a(\overline{G})) + \overline{G}_W\left(rW - f(\mu, a(\overline{G}))\right) + \overline{G}_\mu g(\mu, 2a(\overline{G})) + \frac{\overline{G}_{WW}}{2}\sigma_Y^2 I^2(W,\mu),$$

The existence of such $I(W, \mu)$ follows from properties i) and iii) of h. For times $t \le \tau$ let $a_t = a(\overline{G})$, μ_t be the solution to (19), and W_t be the weak solution to (58), with $I_t = I(W_t, \mu_t)$ and $\{M_t^W\} = 0$. The existence of the weak solution follows from the continuity of actions and volatility I (see Karatzas (1991) 5.4.22). For times $t \geq \tau$, define the processes based off the associated local SSU that gives rise to $(W_\tau, \mu_\tau) \in \partial S$. It follows from the Ito formula that before τ the process $G_t = \overline{G}(W_t, \mu_t)$ follows (58), with $\{M_t^G\} = 0$ and $J_t = \overline{G}_W(W_t, \mu_t) \times I_t$. It follows therefore from Lemma 11 that there is an associated local SSE with marginal benefit of fundamentals at $t = 0$ equal to $\overline{G}(W^*, \mu^*) > G(W^*, \mu^*)$, establishing contradiction. ■

Lemma 13 *At any point where G is twice continuously differentiable, the left-hand-side of equation (21) is weakly lower than the right-hand-side.*

Proof. The proof is closely related to that of Lemma 3. Suppose, by way of contradiction, that there is a point (W^*, μ^*) at which G is twice continuously differentiable and the left-hand-side of (21) minus the right-hand side is at least $2\lambda > 0$. From twice continuous differentiability of G and given that the functions f_μ, f, and g are continuous, there exists a smooth function h defined over a neighborhood S of (W^*, μ^*) such that i) on S the left-hand-side of (21) minus the right-hand side is at least $\lambda > 0$, when applied to $G - h$, ii) $h > 0$ on $intS$, iii) h is strictly convex on S, and iv) $h = 0$ on ∂S. Let $\delta > 0$ be an upper bound on h on S; we may assume that δ is arbitrarily small. Let $\underline{G} := G - h$.

Pick a local SSE that gives rise to (W^*, μ^*) and the marginal benefit of fundamentals above $\underline{G}(W^*, \mu^*)$, at time zero, and let $\{G_t, W_t, \mu_t, I_t\}$ be the processes this local SSE gives rise to (see Proposition 11). From Ito's lemma, the process $D(W_t, \mu_t, G_t) = G_t - \underline{G}(W_t, \mu_t)$ satisfies

$$
\begin{aligned}
\frac{\mathbb{E}\left[dD(W_t, \mu_t, G_t)\right]}{dt} &= \left(r - g_\mu(\mu_t, a(G_t))\right) G_t - \left(I_t + f_\mu(\mu_t, a(G_t))\right) - \underline{G}_W(W_t, \mu_t)(rW_t - f(\mu_t, a(G_t))) \\
&\quad - \underline{G}_\mu(W_t, \mu_t) g(\mu_t, 2a(G_t)) - \frac{\underline{G}_{WW}(W_t, \mu_t)\left[\sigma_Y^2 I_t^2 + d\left\langle M_t^W \right\rangle\right]}{2} \\
&\geq \left(r - g_\mu(\mu_t, a(\underline{G}_t))\right) G_t - \left(I_t + f_\mu(\mu_t, a(\underline{G}_t))\right) - \underline{G}_W(W_t, \mu_t)(rW_t - f(\mu_t, a(\underline{G}_t))) \\
&\quad - \underline{G}_\mu(W_t, \mu_t) g(\mu_t, 2a(\underline{G}_t)) - \frac{\underline{G}_{WW}(W_t, \mu_t)\left[\sigma_Y^2 I_t^2 + d\left\langle M_t^W \right\rangle\right]}{2} - \frac{\lambda}{2} \\
&\geq \left(r - g_\mu(\mu_t, a(\underline{G}_t))\right)\left(G_t - \underline{G}(W_t, \mu_t)\right) + \lambda - \frac{\lambda}{2} > E \times D(W_t, \mu_t, G_t),
\end{aligned}
$$

where $E := \inf\{r - g_\mu\} > 0$, the first inequality holds as long as δ is sufficiently small, and the second inequality follows from properties i) and iii) of h. It follows as in the proof of Proposition 3 that $D(W_t, \mu_t, G_t)$ must exceed δ with positive probability, establishing contradiction. ■